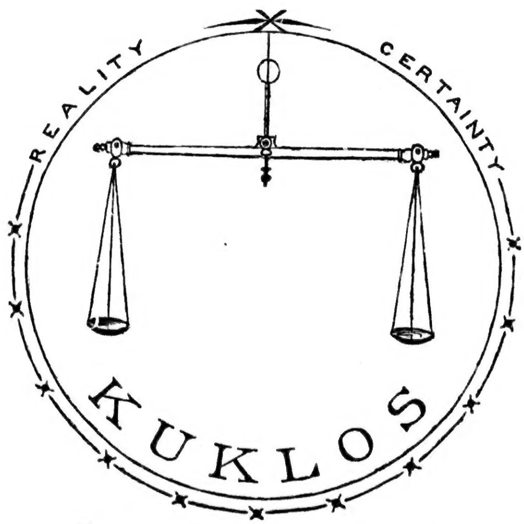
THE CIRCLE

AND

STRAIGHT LINE.



PART SECOND.

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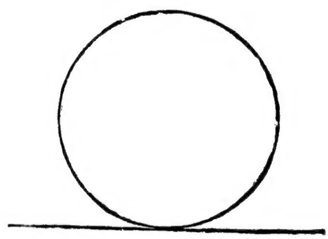

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THE CIRCLE AND STRAIGHT LINE.

'Prove *all* things; hold fast *that* which is *good*.'
St. Paul.

PART SECOND.

The Construction of the Circle.

Having in the preceding exposition of the subject demonstrated 'the true relation of the circle to the straight line' in accordance with Euclid's methodical application of the inductive system of reasoning, we will here give an illustration of the general relationship thereby established, in order—to make, in some measure, apparent the structural completeness and perfection of the circle as a reality, —to show the circumstantial necessity for that particular inter-relationship of the parts which has been now demonstrated to be actually existent, and thereby—to make manifest the great importance of the circle as one of the fundamental facts belonging to the Plan of Creation.

Let us take, in the first place, the construction of Fig. 2, as it was left at page 19, and develop that construction as follows:—

Figs. 10 & 11. CONSTRUCTION. (Fig. 10 is a repetition of Fig. 2 on a smaller scale, similar letters denoting similar parts.) Produce the radius *A.B.* through *A.* and through *B.* and make A.B., (Fig. 11,) the production of *A.B.*, ten times the length of *A.B.* With centre A. and radius A.B. describe the quadrant B.F.; bisect the quadrant at M. and divide the arc M.F. into ten equal parts at the points of equal division *'a. 'b. 'c. 'd. 'e. 'f. 'g. 'h. 'i.* Complete the greater figure similarly to the lesser figure.

Illustration of the fact that the difference between the chord and arc-length of the quadrant is an aliquot part of the chord, and of the arc-length. *

Because R.M. in the greater figure equals *S.R.* and *R.M.* of the smaller figure taken together (to wit, *S.M.*) it follows that, however small the magnitude of a circle may be, the arc-length and the chord of the quadrant of that circle are each divisible into a certain number of equal parts, each of which parts is necessarily equal to the difference of the chord and arc, and each of which parts is an aliquot part of the chord and of the arc-length of the quadrant of any other circle of which the magnitude is an equimultiple of the magnitude of the first circle, however great that other circle may be. Now it is to be particularly observed that, if R.M. of the greater figure were not equal to *S.R.* and *R.M.* of the lesser figure taken together, this would be no longer true. In that case either *S.R.* or *R.M.* of the lesser figure might be evenly divided, but the equal divisional parts would not be aliquot parts of the compound parts belonging to the circle of greater magnitude. The relative magnitude of the greater circle might be such that R.M. of the greater would contain an equimultiple of *R.M.* of the lesser; but, then, S.M. of the same greater circle could not also contain an equimultiple of *R.M.* of the lesser, nor of *S.R.* of the lesser, neither could S.R. of the greater contain an equimultiple of *R.M.* of the lesser, nor, if *S.M.* of the lesser and R.M. of the greater figure were unequal, could R. M. of the greater, however great the magnitude of the greater circle might be, contain an equimultiple of *S.R.* of the lesser.

The general theorem illustrated by this construction may be thus stated. The difference between the chord

* Evidently, in this connection, the relation of the half-chord to the half-quadrant is equivalent to the relation of the chord to the quadrant.

and arc-length of the quadrant of a circle is an aliquot part of the circle; and if a second circle be so related to the first that the magnitude of the first is an aliquot part of the magnitude of the second, then is the difference of the chord and arc-length of the quadrant of the first also an aliquot part of the second circle.

Fig. 12. (*a*) *Construction.*—With centre *A.* and radius *A.B* describe the quadrant *B.S.C.*; draw *B.D.* and *C.D.* and join *A.D.* bisecting the quadrant at *S.* Through *S.* and at right angles to *A.B.* draw *R.S.T.*, intercepting *A.B.* at *R.* and *C.D.* at *T.* Divide *R. S.* into nine equal parts at the points of equal division; from *S.T.*, at the distance *S.X.* equal to one of the equal divisions of *R.S.*, draw *X. X.* perpendicular to *B.D.*, intercepting *B.D.* at *X.* Divide the radius *A.B.* into ten equal parts at the points of equal division. With the point *S.* as a centre and with the radius *S.b.* equal to one of the equal divisional parts of the greater radius *A.B.* describe the small quadrant *b.s.t.*, bisected by the line *A.D.* at the point *s.* Draw the tangent *b.d.* and the sine *S.r.* of the small arc *b.s.*, and join *t.d.*

Scholium. Since the radius of the small arc is one-tenth the radius of the greater arc; the arc, the sine, and the tangent, of the small arc are respectively one tenth of the arc, of the sine, and of the tangent, of the greater arc; and therefore, also, the *difference* of the *sine and arc* on the small scale is one-tenth of the *difference* of the *sine and arc* on the larger scale (*i.e.*—*x.s.* is the one-tenth of *X.S.*) But *s.t.* the *difference* of the *sine and tangent* on the small scale is manifestly one-tenth of *S.T.* on the large scale, and *x.t.* must be also necessarily one-tenth of *X.T.* It will follow, if the small arc *b. s.* is straightened upon or rolled upon the line *b.d.*, that the point of contact must fall upon the line *X.X.* This will become more apparent by developing the construction.

Prop. (in quantitive geometry.)

THEOREM.—*The quantity of length by which the length of the half-quadrant of a circle exceeds the length of the sine of the half-quadrant is equal to the one-tenth part of the length contained in the half-quadrant.*

Construction.—Fig. 12. *(a) has been already described.*

Specification.—Let *B.S.* be the half-quadrant of which *S.R.* is the sine, and let *S.X.* indicate the diff. in length between *B.S.* and *S.R.*;—*S.X.* shall equal one of the divisional parts of the arc *B.S.*, if *B.S.* be divided into ten equal parts.

Demonstration.—Because the radius *S.b.* is the one-tenth of the radius *A.B.*, the sine *S.s.* of the lesser arc *b.s.* is one-tenth of *R.S.* the sine of the greater arc. But *S.X.* the diff. of the arc-length and sine of the greater arc is ten times greater than the similar diff. *s.x.* of the arc-length and sine of the lesser arc; therefore *s.x.* + *S.s.*, that is *S.X.*, is the one-tenth of *R.X.* Now *R.X.* is the arc-length of *B.S.*; and *S.X.* is the diff. between that arc-length and the sine of the same arc. Wherefore the theorem has been demonstrated.

Scholium.—The foregoing prop. may be numerically exhibited in units of radius: for example:

The radius *A.B.* = 10.—The sine *R.S.* = $\sqrt{50}$ = 7·071068
* Therefore *R. X.* = 7·856742
" " *S.d.* = 1.—The sine *S.s.* = $\sqrt{\frac{1}{2}}$. = 0·707107
S X. = 0·7856742
s.x. = 0·0785674

Development of the Construction. Fig. 12. *(b.)* From the point *X.* on the line *B.D.*, at the distance *X.V.* equal to *X.D.*, take the point *V.* From *V.* at right angles to *S.D.* draw *V.F.*, intersecting *S.D.* on the line *X.X.*, and intersecting the line *b.d.* at *v.*

Illustration.—Now, if a second arc similar and equal to the greater arc *B.S.C.* be described in such wise that the point *V.* become the extremity of the secant, and the

* Because *R.X.* = *R.S.* divided by 9, × 10.

production of *V.F.* be the secant, and the production of *V.D.* be the tangent, of the second greater arc, it is manifest that the second arc, so described, must, necessarily, intersect the first greater arc on a production of the line *X.X.* Now *S.d.*, the secant of the small arc, is included in the secant *A. D.* of the first greater arc, and likewise, if a second small arc be described similarly related to the second greater arc, then will the secant of the second lesser arc be included in the (production of the) line *V.F.*, the secant of the second greater arc; therefore since *x.d.* is one tenth of *X.D.*, *x.v.* must necessarily be the one tenth of *X.V.* It becomes evident accordingly that the point *x.* of the lesser arc necessarily falls upon the line *X.X.*

Fig. 12. *(b.)* may be further developed by completing the double figure. Produce *X.T.* through *T.* and on that production make *(R.)X.* equal to *R.X.* Produce *A.C.* through *C.* and produce *B.D.* through *D.* and make *C.(A.)* and *D.(B.)* each equal to *T. (R.)* on the line *R.(R).* With centre *(A)* and radius *(A.)(B.)* describe the arc *(B.)(C.)* intersecting the arc *B. C.* in the line *X.X.* With the centre *F.* (on the line *R.(R.)* and radius *F.z.* equal to *S.b.* (equal to *A.B.* divided by 10), describe the arc *z.y.* Join *z.d.b.* Draw *n.m.* the sine of the arc. Join *y.v.* and produce *F.z.* through *z.* intercepting the line *B.(B.)* at *Q.*

The analytical value of the construction thus developed may be understood by considering that the line *R.(R.)* contains the sines *R.S.* and *(R.)F.* of both the arcs together with the space *S. F.* And *R.(R.)* also contains the tangents *R.T.* and *(T.)(R.)* of both the arcs, *less* *T.(T.)* Again, because the radius *F.z.* is equal to the radius *s.b.*, and because *F.z.* has the same relation to the arc *(B.)(C.)* that *S.b.* has to *B.C.* therefore *y.x.* is equal to *t.x.*, &c., &c.

Example. The radius *A.B.* = 10.

R. (*R.*) = 2 *R.T.* − 2 *X.T.* equals 20 − 4·28652 = 15·7134

R. (*R.*) = 2 *R.S.* + 2 *S.X.* equals . . 14·14214 + 1·57134 = 15·7134

R. (*R.*) = 2 *R.T.* + 2 *S.X.* − 2 *S.T.* equals 20 + 1·57134 − 585776 = 15·7134

R. (*R.*) = *R.X.* + *R.X.* Which equals

$$\left.\begin{array}{l} R.\ S. + \dfrac{R.S.}{10} + \dfrac{R.S.}{100} + \dfrac{R.S.}{1000} + \&c. = 7{\cdot}85674 \\ (R.)F. + \dfrac{(R.F.}{10} + \dfrac{(R.F.}{100} + \dfrac{(R.F.}{1000} + \&c. = 7{\cdot}85674 \end{array}\right\} 15{\cdot}7134$$

Illustration by the construction. Because the line *x.x.* belonging to the small arc is included in the line *X.X.* of the greater, the line *S. X.* is equal to *S. s.* added to *s.x.* (i.e. the *diff.* of the *sine and arc* on the large scale is equal to the sine of the arc added to the *diff.* of the *sine and arc* on the small scale.

Quantitive and Numerical Illustrations, to Fig. 12.

R.T. = A.D. =	10·000000	S.D. = A.D. — A.S. =	4·142135
R.S. = $\sqrt{50}$ =	7·071068	S.T. = R.T. — R.S. =	2·928932
A.D. = A.S. + S.D. =	14·142135	R.S. = R.S. + $\frac{\text{R. S.}}{10}$ =	7·778174

$$\text{R.}x. = \frac{\text{R.S.}}{9} \times 10 = 7{\cdot}856742$$

Because $\text{S.s.} = \dfrac{\text{R.S.}}{10}$ And $\text{S.t.} = \dfrac{.\text{T.}}{10}$

And $\text{S.s} + \text{s.t.} = \dfrac{\text{R.T.}}{10}$ And R.T. = R.S. + S.T.

Therefore (1) R.s. + s.t. = R.t.

(2) R.T. − 9 S.t. + R.S. = R·t.

(3) R.S. + S.t. = R.t.

And

(1) 7·7781748 + ·292893 = 8·0710678

(2) 10 − 9 + 7·0710678 = 8·0710678

(3) 7·0710678 + 1·0000 = 8·0710678

Because $S.\ s. = \frac{R.S.}{10}$ Therefore $R.s. = R.S. + \frac{R.S.}{11}$ and consequently if the scale be again reduced to one tenth of the lesser figure, and then again reduced to one tenth of the last, and so an 'ad infinitum' $R.\mathfrak{X}.$ must evidently include the (sum of the one-tenths) one-tenth of each and every of all the sines; (i. e., the one-tenth of each of the sines of all the figures from the grestest (R.S.) to the least imaginable). Therefore $R.\mathfrak{X}.$ equals

$$R.S. + \frac{R.S.}{10} + \frac{R.S.}{100} + \frac{R.S.}{1000} + \frac{R.S.}{10000} + \text{\&c., ad inf.}$$

7·0710678
 70710678
  70710678
   70710678
    70710678
     70710678
&c., &c., &c., &c., ad infinitum.

7·856742 &c., &c.

Arithmetical Illustration of the fact that x. falls on the line $\mathfrak{X}.\mathfrak{X}.$

Because $S.s. = \frac{R.S.}{10}$ Therefore $S.s. = \frac{S.R. + S.s.}{11}$

$$\cdot 70710678 = \frac{7\cdot 0710678}{10} = \frac{7\cdot 0710678}{11} + \frac{\cdot 70710678}{11}$$

And $S.s : S.x. :: S.R : R.\mathfrak{X}.$ Therefore.

$S.x. = S.s. + \frac{S.s.}{9}$ $\quad \cdot 7856742 = \cdot 70710678 + \cdot 07856742$

$R.\mathfrak{X}. = S.\mathfrak{X}. + R.S.$ $\quad 7\cdot 856742 = \cdot 7856742 + 7\cdot 0710678$

$S.s. = \frac{7\cdot 77817458}{11} = \cdot 70710678$

But S.s. : S.x. :: S.R. : R.$\mathcal{X}$.

$$S.\ s. = \frac{S.R.}{10} \quad R.\ S. = S.R. + S.s. = 7{\cdot}778174$$

$$S.\ \mathcal{X}. = S.s. + \frac{S.\ s.}{9} = {\cdot}7856742 = \frac{R.\mathcal{X}.}{10}$$

$$S.\ \mathcal{X}. \times 10 = 7{\cdot}85674 = R.\mathcal{X}. = S.\mathcal{X}. + S.R.$$

$$s.\ x. = \frac{S.\ x.}{10} \text{ And } R.x. = R.s. + s.x.$$

That is, $7{\cdot}778174 + {\cdot}078567 = 7{\cdot}856742$

And again $\mathcal{X}$.T. = R.T. − R.$\mathcal{X}$. = 2·143258

$$\mathcal{X}.t. = S.t. - S.\mathcal{X}. = 2143258 \text{ (i. e. } \mathcal{X}.t. = \frac{\mathcal{X}.T.}{10})$$

Because the lesser (right angled) triangle S.W.D is similar to the greater triangle A.R.S., therefore

$$\frac{S.W.}{9} : S.D. :: \frac{R.S.}{9} : A.S.$$

$$\text{And } \frac{S.W.}{9} \times \frac{A.S.}{S.D.} = \frac{S.R.}{9}$$

$$\text{That is, } \frac{2{\cdot}928932}{9} \times \frac{10}{4{\cdot}14213} = \frac{3{\cdot}25437}{4{\cdot}14213} = {\cdot}7856742$$

Quantitive and Numerical Illustration (demonstration) *of the fact* that the quantity obtained by analytical methods, which is now supposed to represent the ratio of the circumf. of a circle to the diameter is erroneous.

Scholium. The quantity under examination, stated as the ratio of the circle to a unit of diameter, equals 3·14159; therefore, taking the radius = 10. the arc of 45 degrees = 7·85397.

By Fig. 12. Taking the radius A.B. = 10.

And assuming the arc-length to be *as stated*, then

R.$\mathcal{X}$. = 7·85397 R.T. = A.B. = 10

R.S. = $\sqrt{50}$ = 7·071067 And therefore :—

S.$\mathcal{X}$. = R.$\mathcal{X}$. − R.S. = ·78291

$$\text{Again, } S.\ s. = \frac{R.\ S.}{10} = {\cdot}70710678$$

$$\text{And, } s.\ \mathcal{X}. = \frac{S.\mathcal{X}.}{10.} = {\cdot}078291$$

Therefore S.$\mathcal{X}$. = S.s. + s.$\mathcal{X}$. = ·78539.

But it has been shown that S.$\mathcal{X}$. = ·78291, and therefore the same line has (appears to have) two different lengths which is impossible.

And again, further :—

$$\text{R.s.} = \text{R. S.} + \frac{\text{R.s.}}{10.} = \text{S.s.} \times 11 = 7{\cdot}7781745,$$ consequently (R.$\mathcal{X}$. – R.s.) × 10 = s.$\mathcal{X}$. × 10 = S.$\mathcal{X}$. = ·75795.

But it has been already shown that S.$\mathcal{X}$. equals ·78539, and also equals ·78291—and therefore the same line S.$\mathcal{X}$. is (apparently) demonstrated to have three different lengths, which is absurd.

Again by the assumption ; R.$\mathcal{X}$. = 7·85397

$$\text{But S.}\mathcal{X}. = \frac{\text{R.}\mathcal{X}.}{10} = {\cdot}785397$$

R.$\mathcal{X}$. = S.$\mathcal{X}$. + S.R. = ·785397 + 7 0710678 7·85646.

Therefore R.$\mathcal{X}$. has (appears to have) two different lengths which is impossible. Wherefore the assumption is erroneous.

Fig 13.

Construction.—Describe the quadrant *B.C.* bisected by the line *A.D.* in the point *S.* Draw the sine *R.S.* and the tangent *B.D.* of the arc *B.S.* Join *A.C.* and *D.C.* and produce *R.S.* through *S.*, intercepting *D.C.* at *T.* Divide the radius *A.B.* into ten equal parts at the points of equal division. With centre *S.* and radius *S.b.*, describe the quadrant *b.t.* bisected by *S.D.* in the point *s.* ; draw the sine *s.r.* and the tangent *b.d.* of the arc *b.s.*

Produce the line *B.D.* through $\mathcal{X}$., and make $\mathcal{X}$.*F.* equal to $\mathcal{X}$.*B.* ; produce *A.C.* through *C.*, and from *F.* draw *F.E.* perpendicular to *B.F.*, and intercepting the production of *A.C.* in the point *E.* From the point *n.*, where *A.D.* intersects $\mathcal{X}$.$\mathcal{X}$., join *n.B.*, and from the same point join *n.F.* intersecting *C.D.* at *H.*, and from

the same point draw also *n.I.E.* at right angles to *A.D.* intersecting *C.D.* at *I.*, and intercepting the point *E.* at the vertex of the angle *A.E.F.* With centre *I.* and radius *I.H.* describe the quadrant *H.K.* bisected by *I.n* in the point *e.* Draw the sine *G.e.* of the arc *H.e.*, and from *d.* through *e.* draw *d.t.* perpendicular to *R.T.*, and intercepting *R.T.* at *t.* From *K.* at the extremity of the quadrant *H.K.* draw *K.m.* perpendicular to *R.T.* and intercepting *R.T.* at *m.*

Scholium.—We have now three similar triangles, namely : *A.n.B.*, *S.n.b.*, *I.n.H.* Because *n.* is a point in the line *X.X.*, and *T.* is a point in the radius *I.H.*, and *R.* a point in the radius *A.B.*; the ratio of the radius *I.H.* to the radius *A.B.* is the same as the ratio of the line *X.T.* to the line *X.R.*, and also the ratio of the radius *S.b.* to the radius *I.H.*, the same as the ratio of *S.X.* to *T.X.*

Illustration (a) of the fact that the point x. falls on the line X.X.

Because the radius *S.b.* is the one-tenth of the radius *A.B.*, therefore *s.x.* is the one-tenth of *S.X.* and *x.d.* the one-tenth of *X.T.* But *S.t.* (*i.e.*, *S.x.* + *x.d.*) is the one-tenth of *R.T.*, and the ratio of *S.x.* to *x.t.* is the same as the ratio of *R.X.* to *X.T.* (because the ratios of similar arcs each to each, and of the tangents, and also of the sines of similar arcs each to each, are directly as the ratios of the radii each to each, to which they respectively belong.) And the ratio of *R.S.* to *T.t.* is also the same as the ratio of *R.X.* to *X.T.*, therefore *x.T.* is equal * to *X.T.* and *x.R.* equal to *X.R.* Wherefore it becomes evident that the point *x.* is included in the line *X.X.*

By the same reasoning applied to the secant,

Because *S.b.* is the one-tenth of *A.B.*, therefore *S.d.* is the one-tenth of *A.D.*, and *S.s.* the one-tenth of *R.S.*, and *s.d.* the one-tenth of *S.D.* But *s.n.* is, therefore, the one-tenth of *S.n.*, and *n.d.* the one-tenth of *n.D.* Consequently the point *n.*, which is a point in the line

* Because $x.T. = x.t. \times 10$, and $x.R. = x.r. \times 10$.

$x.x.$, is also included in the line $\mathfrak{X}.\mathfrak{X}.$ Wherefore it clearly appears that the line $x.x.$ is the same line as $\mathfrak{X}.\mathfrak{X}.$

Demonstration (a) by the construction Fig. 13 (to prop. B.) *That the diff. of the sine and arc-length of the half-quadrant is one tenth the length of the half-quadrant.*

Because $\text{S.t.} = \dfrac{\text{R.T.}}{10}$ $\qquad s.\mathfrak{X}. = \dfrac{S.\mathfrak{X}.}{10}$

$$\text{R.T.} - \text{S.t.} = \frac{9\ \text{R.T.}}{10} = \text{R.S.} + \text{T.t.}$$

and $\text{R.T.} = \text{R.S.} + \text{T.t.} + \text{S.t.} = (\text{R.S.} + \text{S.}x.) + (\text{T.t.} + \text{t.}x.)$

and also $\text{R. T.} = \text{R.S.} + \text{T.t.} + \dfrac{\text{R.S.}}{9} + \dfrac{\text{T.t.}}{9}$

Therefore $\text{S.}\mathfrak{X}.$ equals $\dfrac{\text{R.}\mathfrak{X}.}{10}$ And $\text{t.}\mathfrak{X}.$ equals $\dfrac{\text{T.}\mathfrak{X}.}{10}$

For let it be supposed possible that $\text{S.}\mathfrak{X}.$ may be a magnitude less than $\dfrac{\text{R.}\mathfrak{X}.}{10}$ then must $\text{t.}\mathfrak{X}.$ be a magnitude greater than $\dfrac{\text{T.}\mathfrak{X}.}{10}$ (and $s.\mathfrak{X}.$ less than $\dfrac{S.\mathfrak{X}.}{10}$)

But R.T. is wholly compounded of $\text{R.}\mathfrak{X}.$ and $\text{T.}\mathfrak{X}.$ together. (Demonst. b), and R.T. contains R.S. and T.t. together with $\dfrac{\text{R.T.}}{10}$ (i.e. $\text{R.S.} + \text{T.t.} + \dfrac{\text{R.S.}}{9} + \dfrac{\text{T.t.}}{9}$) and $\dfrac{\text{R.T.}}{10} - \dfrac{\text{R.S.}}{9} = \dfrac{\text{T.t.}}{9}$ consequently if $\text{S.}\mathfrak{X}.$ be any magnitude less than $\dfrac{\text{R.S.}}{9}$ then must the remaining magnitude, of which, together with $\dfrac{\text{R.S.}}{9}$, the magnitude $\dfrac{\text{R.T.}}{10}$ is compounded, be greater than $\dfrac{\text{T.t.}}{9}$

Now T.t. is the sine of an arc similar to the arc of which R. S. is the sine, and therefore it is impossible that the ratio of $\text{t.}\mathfrak{X}.$, the diff. of the sine and arc length

of H.e., to T.t., the sine of the arc H.e., can be greater than the ratio of S.𝔛., the diff. of the sine and arc-length of B.S., to R.S. the sine of the arc B.S. Wherefore it is demonstrated that S.𝔛., the diff. of the sine and arc-length of the half quadrant B.S., is the one-tenth part of the arc-length contained in the half quadrant B.S. Q.E.D.

Demonst. (b). *That the line R.T. is wholly compounded of the arc-length of B.S. together with the arc-length of H.e., and that the same point x. is the extremity of the arc-length of each of the two arcs—to wit, of the arc B.S. and the arc H.e.*

(If the arc-length of B.S. contained between R. and a point indicated by 𝔛. in the line R.T. be taken from R.T. it is manifest that a magnitude will remain which must be equal to the arc-length of some arc contained between T. and the point indicated by 𝔛.) The triangle I.H.n. is similar to the triangle A.B.n. and the bases of the two triangles together—to wit H.n. and B.n. together—include all of the magnitude contained in the line R.T., and, since 𝔛, is the divisional point which divides the part of the line R.T. belonging to R, from the part thereof belonging to T, the ratio of R.𝔛. to T.𝔛. must be the same as the ratio of B.n. to H.n. and the same as the ratio of the radius A.B. to the radius I.H. Therefore the point 𝔛 has a similar relation to each of the arcs, and to the similar lines belonging to each of the arcs respectively —to wit, to the arc H.e. and the arc B.S. . . . to the sine G.e. and the sine R.S. . . . to the 'diff. t.x, of the sine and arc-length' of the arc, H. e. and the 'diff. S.x. of the sine and arc-length' of the arc B.S. . . to the 'diff. of the arc-length and tangent' x.m. and the 'diff. of the arc-length and tangent' x.T. Now, if the difference between the sine R. S. and the arc length of the arc B.S. were either less or greater than S.𝔛. then would x.m. be not in the same ratio to x.T. as the ratio of 𝔛.T. to 𝔛.R. (for if it be supposed possible that

the tangent of the arc H.e. may be greater than T.m. then will t.m. no longer have the same ratio to T.t. which S. T. has to R. S.; nor will the ratio of t.m. to S.T. be the same as the ratio of the radius I.H. to the radius A. B.). Therefore, since the sine R. S. and the *diff.* S.$\mathcal{X}$., of the sine and the arc, together with the diff. $\mathcal{X}$.T. of the arc and the tangent belonging to B.S. wholly compound the line R.T., and since T.t. is to t.x. and to x.m., respectively, in the same ratio as R.$\mathcal{X}$. is to S.x., and x.T. respectively, the point x. at the extremity of the arc-length of each arc cannot be other than the same point $\mathcal{X}$. which divides the line R.T. in such wise that the part R.$\mathcal{X}$. thereof has the same ratio to the part T.$\mathcal{X}$. which the radius A.B. has to the radius I.H. *Wherefore it is demonstrated*, &c., Q.E.D.

Quantitive and numerical illustrations to the construction. *Fig.* 13.

By the Construction :—

$$\text{R.T.} = \text{A.B.} = 10{\cdot}00000$$
$$\text{R.S.} = \sqrt{50} = \underline{7{\cdot}07107}$$
$$\text{R.T.} - \text{R.S.} = \text{S.T.} = 2{\cdot}92893$$
$$\text{S.t.} = \text{S.b.} = \frac{\text{A.B.}}{10} = \underline{1{\cdot}00000}$$
$$1{\cdot}92893 = \text{T.t.}$$

By Demonstration :—

$$\text{S.}\mathcal{X}. = \frac{\text{R.S.}}{9} = \frac{7{\cdot}07107}{9} = 0{\cdot}785674$$

$$\text{R.}\mathcal{X}. = \text{RS.} + \text{S.}\mathcal{X}. = 7{\cdot}07107 + 0{\cdot}785674 = 7{\cdot}85674$$

Now $t.x.$: T.t. : : S.$\mathcal{X}$. : R.S.

$$\text{Therefore } \frac{1{\cdot}92893 \times {\cdot}785674}{7{\cdot}07107} = {\cdot}214325 = \text{t. } x.$$

$$\text{T.}x. = \text{T.t} + \text{t.}x. = 1.92893 + {\cdot}214325 = 2{\cdot}14325$$

$$\text{And T.}\mathcal{X}. = \text{S.T.} - \text{S.}\mathcal{X}. = 5{\cdot}92893 - {\cdot}7856742{\cdot}14325$$

Test :

Let it be possible for S.𝔛. to be less than $\frac{R.S.}{9} = \frac{\cdot 707107}{9}$ and let S.𝔛. $= 0{\cdot}78283$ Then

$$R.𝔛. = R.S. + S.𝔛. = 7{\cdot}07107 + 0{\cdot}78283 = 7{\cdot}8539$$

Now $t.x. : T.t. :: S.𝔛. : R.S.$

$$\text{Therefore } \frac{1{\cdot}92893 \times \cdot 78283}{7{\cdot}07107} = \cdot 213549$$

and $T.𝔛. = T.t. + t.x = 1{\cdot}92893 + \cdot 213549 = 2{\cdot}14248$

But $T.𝔛. = S.T. - S.𝔛. = 2{\cdot}92893 - \cdot 78539 = 2{\cdot}14354$

Therefore the same line T.𝔛. has two different quantities of magnitude which is impossible. In the same manner it may be shown that S.𝔛. cannot be greater than $\frac{R.S.}{9}$ Wherefore $S.𝔛. = \frac{R.S.}{9}$

Illustration of the relationship of the arc-length to the sine of the half-quadrant. By the construction, Fig. 13.

It is evident that, since $s.x. : S.x :: s.b. : A.B.$, if another arc be described with the point *s.* as a centre, and a radius *s.c.* less than *s.b.* in the same ratio that *s.b.* is less than *A.B.*, then *s.x.* must contain the sine and arc-length of the arc so described with the radius *s. c.*; and, again, the diff. *f.x*, between the sine and arc-length of this small arc must contain the sine and arc-length of a smaller arc, described with the point at the extremity of the sine of the last small arc as a centre, and with a radius less than *s.c.* in the same proportion that *s.c.* is less than *s.b.*; and, in the like manner, arcs may be continually described, each arc being less than the arc preceding it in the same proportion, so long as there be any assignable quantity of distance remaining between the extremity of the sine of the arc last described and the point *x*.

Note.—The reader may, if he please, describe Fig. 13, on a scale ten times larger; the small arc and radius, *b.s.* and *S.b.* will be then enlarged to the size of the

greater arc and radius—*B.S.* and *A.B.*, and the second smaller arc, described with radius *s.c.*, will be then, if described, the same size and occupy the same relative position in the enlarged figure, which the arc *b.s.*, described with the radius *S.b.*, occupies in our Figure 13. The figure may be then again enlarged in the same proportion as before and a third smaller arc be described; and so on 'ad infinitum.' Instead, however, of actually describing enlarged figures, an alteration in the letters denoting the parts will serve to illustrate the case, by supposing the lesser arc to have been enlarged into the greater arc, as often as may be desired.—

Quantitive and Numerical Illustration.

$$\text{R.}\mathfrak{X}. + \text{T.}\mathfrak{X}. = \text{R.T.} = 10.0000.$$

Since R.$\mathfrak{X}$. equals R.S. $+ \dfrac{\text{R.S.}}{10} + \dfrac{\text{R.S.}}{100} + \dfrac{\text{R.S.}}{1000} +$ ad inf.

And T.$\mathfrak{X}$. equals T.t. $+ \dfrac{\text{T.t.}}{10} + \dfrac{\text{T.t.}}{100} + \dfrac{\text{T. t.}}{1000} +$ ad inf.

$$\text{Therefore} \left\{ \begin{array}{l} 707107 \\ \quad 707107 \\ \qquad 707107 \\ \qquad\quad 707107 \\ \qquad\qquad 707107 \\ \qquad\qquad\quad 707107 \\ \quad \text{\&c., \&c.,} \\ \text{`ad infinitum.'} \end{array} \right\} + \left\{ \begin{array}{l} 192893 \\ \quad 192893 \\ \qquad 192893 \\ \qquad\quad 192893 \\ \qquad\qquad 192893 \\ \qquad\qquad\quad 192893 \\ \quad \text{\&c., \&c.,} \\ \text{`ad infinitum.'} \end{array} \right\} = 10.0000$$

Figure 11.

Quantitive and Numerical Illustrations to the construction.

(By the construction :—A.B. the radius of the greater figure (Fig. 11), equals *A.B.* the radius of the lesser figure (Fig. 10) multiplied by ten.)

By the established trigonometrical relationship of the parts:

	In the greater Fig. 11.	In the lesser Fig. 10.
The radius	A.B. = 10·000000	*A.B.* = 1·0000000
The sine $\sqrt{50}$ S.R. =	7·071068	*S.R.* = 0·7071068
The diff. of the arc and sine	$\frac{\text{S.R.}}{9}$ = M.R. = 0·785674	*R.M.* = 0·0785674

The arc length M.F. or S.M. = 7·85674

The arc length *M.F.* or *S.M.* = 0·785674

Therefore:

$$\text{R.M.} = \frac{\text{S.M.}}{10} = 0{\cdot}785674 \qquad S.M. = 0{\cdot}785674$$

DEMONSTRATION *that R.S., the sine of the arc O.S. (Fig. 2.) contains a certain number of equal divisional parts, each of them equal to R.M. the diff. of the sine and the arc-length.*

Divide the radius *A.M.*, of the arc *B.M.*, at *I.* the point of bisection. With centre *I.* and radius *I.M.*, describe the quadrant *H.M.G.* half the length (magnitude) of the quadrant *B.M.F.* (because the radius *I.M.* is half of *A.M.*) The point *M.* which bisects the greater, bisects also the lesser quadrant, and the arc *M.G.*, the half of the lesser, is similar to *M.F.*, the half of the greater quadrant.

Now, let the arc *H.M.G.* be rolled upon the straight line *H.W.* until the point *M.* becomes in contact on the line . . . Since the arc is similar to the greater arc and the motion is similar, and the lesser contains one half the length of the greater, it is manifest that the point *G.* at the extremity of the arc will intersect the line *M.S.* at the point of bisection of that line . . to wit, at the point *e.*, half the distance of *S.* from *M.* on the line *M.S.* But since the two arcs are similar and the magnitude of the lesser is one-half of the greater, the sine of the lesser is also one-half the sine of the greater. Consequently the remaining half *e.S.* of the line *M.S.* must contain one-half of

the sine-length and one-half of *R.M.* the difference between the sine and arc-length of the greater arc. If, therefore, the sine *S.R.* of the greater arc be evenly divided, *e.S.* contains one-half of those even divisions, together with one-half of R.M. But *e.S.* also contains one-half of the even divisions contained in the line *M.S.* (arc-length.) Therefore, each of the even divisions of the sine *S.R.* is an aliquot part of the line *M.S.*, and consequently *R.M.* is an aliquot part of *R.S.*

But the line *S.M.* is divided into ten equal parts; and *R.M.* is known to contain either the whole or very nearly the whole of one of the divisional parts. If *R.M.* were equal to the half of one of the ten divisional parts, then would *S.R.* be equal to nine and a half of those parts; that is, $\frac{S.R.}{19}$ would then equal *R.M.* But since *R.M.* is known to equal considerably more than the half of one of the parts and is shown to be an aliquot part of *S.R.*, it must equal the whole of one of those parts. Therefore, since *R.M.* is an aliquot part of *S.R.*, *R.M.* must be one of the ten equal divisional parts of *S.M.*; and *S.R.* (equal to *S.M.—M.R.*) must evidently contain nine of those equal divisions. Wherefore, the sine *R.S.* is shown to contain nine equal divisional parts each of them equal to *R.M.* the diff. of the sine and arc-length.

(Fig. 14.)—CONSTRUCTION.

With centre *A.* and radius *A.B.* describe the quadrant *B.M.F.* Bisect the quadrant in the point *M.*, and through *M.* draw *A.K.*, the secant. Draw, also, *M.N.*, the sine of the arc *B.M.*, intercepting *A.B.* at *N.* Divide *A.N.* into ten equal parts, and divide also *N.B.* into ten equal parts, at the points of equal division, 1.2.3. 4.5.6.7.8.9., respectively:—

With radius 9.9. describe a quadrant of one-tenth less magnitude than *B.M.F.*

With radius 8.8. describe a quadrant of two-tenths less magnitude than *B.M.F.*

With radius 7.7. describe a quadrant of three-tenths less magnitude than *B.M.F.*

And so on. Finally :—

With radius 1.1. describe a quadrant of nine-tenths less magnitude than *B.M.F.*

To each successive arc draw the tangent and cotangent of the arc, and also from each of the points in the line *M.N.*, where the line is intersected by the successive arcs, draw a perpendicular through the line *B.K.*

We now obtain three distinct divisions of *B.K.*;

(1) the successive tangent-lengths dividing *B.K.* into ten equal parts;

(2) the successive arc-lengths dividing *B.O.* into ten equal parts;

(3) the successive sine-lengths dividing *B.D.* into ten equal parts.

Taking the radius *A.F.* = 10. The quantitive relations of the three lines are *B.K.* = 10·000000

B.O. = 7·856742 *B.D.* = 7·071068

And these relative proportions are precisely the same in the least as in the greatest of the arcs, and however far the reduction in size may be carried, so long as it be possible to describe a similar arc, or to imagine a similar arc to be described, these lines pertaining to the arc, and which even in imagination are inseparable therefrom, must necessarily have the same relative proportions each to each and each to the arc. Referring to Fig. 11, we observe that so soon as the quadrant *B.M.F.* commences to roll upon the line *B.E.*, the (upper half quadrant) arc *M.F.* commences at *M.* to pass through the line *M.S.*; when the rolling process is completed, and the point *M.* becomes in contact on *B.E.*, the entire arc has passed through the line *M.S.*, and every component part of the arc has passed through the line at the same angle; therefore if we suppose a number of perpendiculars drawn through the line *M.S.* at any very

minute distances from each other, each proportionally minute part of the arc in passing through the line *M.S.* would form with that line and the perpendicular (which cuts the line where the minute portion of the arc commences to pass through,) the figure *R.O.S.* on a scale proportionate to the minute magnitude of the lines compounding it. Now when the point *M.* of the arc has arrived at *O.* the straight * longitudinal extension of the arc is contained in the length of the line—to wit, in *(O.T.) R.S.* Every portion of the arc, therefore, relatively to the straight longitudinal measurement thereof on the line *O.E.* suffers a diminution in length in the ratio of *R.S.* to *M.S.* Consequently a manifest relationship becomes apparent between the horizontal and perpendicular lines *M.S.* and *R.O.*, which must be so proportioned to each other as to result in the arc length *O.S.* containing as its sine the horizontal length *R.S.* The quantitive relationship expressed in figures makes this relationship readily apparent,

for let *A.B.* the radius, = 10, then :—

M.S. the arc-length, = 7·85674

R.S., the sine, = 7·07107

R.O. the versed sine = 2·92893

Now *R.S.* + *R. O.* = *A. B.*

thus, ·7071068 × 10 = 7·071068.

$$\cdot 7856742 - \frac{\cdot 7856742}{10} \times 10 = 7\cdot 071068$$

And 7·071068 + 2·92893 = 10·00000

The quantitive relationship of the other principal lines which occur in these figures (Figs. 11 and 14) may be briefly noticed. The secant *A.K.* is equal to the chord

* We mean by this expression—the longitudinal space between two perpendiculars, the one drawn through the point at one extremity of the arc, and the other perpendicular drawn through the opposite extremity of the arc.

of the quadrant which, relatively to the radius * as 10, contains $\sqrt{200} = 14.14214$ (equal to twice the sine of the arc *B.M.*) and, therefore, if divided into nine equal parts, the quadrant contains ten equal parts, each equal to each of those nine parts (Coroll. prop. *D.*) Again, the difference of the radius *A.M.* and secant *A.K.*, namely *M.K.*, is in the same proportion to *M.D.* as *A.M.*: *A.N.*, to wit, as 10: 707107. Therefore, *M.K.* = 4·14214. (*i. e.* $\sqrt{200} - 10$). *M.D.* = 2.92893. . . The numerical values of these quantities of magnitude in units of radius exhibit their relationship, thus: *R.S.* + *R.O.* = *A.B.* (or *B.K.*) 7·07107 + 2·92893 = 10

$$\frac{M.\ K.\ \times\ M.\ N.}{B.\ K.} = M.\ D.$$

$$4{\cdot}14214 \times \frac{7{\cdot}07107}{10} = 2{\cdot}92893.$$

MATHEMATICS AND THE ART OF COMPUTATION.

The fundamental character of the relation of the circle to the science of number and quantity is established by demonstration that the difference of the quadrant and the chord of the quadrant, (of the arc and sine of the arc) is an aliquot part of the quadrant and of the chord, and that the number of those equal parts contained in the chord being nine—the quadrant contains ten; because herein we find conclusive evidence that the (so-called) Arabic system of notation is not an artificial human contrivance, but a great natural fact of a primary character, a fundamental part of the Science of

* It may be observed that taking the radius = 1. The chord of the quadrant becomes $\sqrt{2}$ and the sine of the half quadrant becomes $\sqrt{\frac{1}{2}}$.

Creation. When this is well understood it will be only necessary, in order to appreciate in some measure the immense number of facts of a secondary character, belonging to the science of Number and Quantity, furnished by the correlation of the lines compounding (or belonging to) the circle, to consider the relationship of the few primary lines—exhibited in their numerical values furnished above—in connection with the method shown in Fig. 14, of reducing the arc *B.M.* from a radius equalling 10. to the one-tenth thereof by a gradation of similar arcs, through the nine successive intermediate magnitudes. It is to be noted that each of these lesser arcs has its own dependent lines with the same respective proportions each to each as the lines belonging to the greater arc, and that since each lesser arc has a known definite ratio to the greater arc, each* and every line belonging to each lesser arc has a known definite quantitive (numerical) ratio to each and every line belonging to the greater arc. Again, this primary division of the arc of given magnitude (that of the arc with radius = 10 into units,) may be subjected to subdivision, and then, to further subdivision; each division turnishing an additional series of quantitive representatives or members, each member of the series having its own system of compounding lines, with their definite ratios each to each, and each to the radius of that member, and each member also having a definite known ratio to each of the other members of that series, and, through the primary member of that series, having a definite known ratio to the general primary,—that is, to the primary arc or circle of given magnitude; and, through the general primary, having also, a known definite ratio to each of the members, and to each and all the definite divisions and subdivisions of the primary circle.

* Or which may be readily known.

Now these relations of the compounding lines belonging to the circle are natural facts. They may be justly considered the *material* out of which and with which the instruments of quantitive analysis are to be constructed, to be improved, to be simplified, and to be, as far as possible, perfected.

That these facts are existent, and that they are not by any means deeply buried, is well known to many of those whose occupation it is to *till the ground*, or whose especial duty it is so to do; but many of those persons prefer, with a strange, and, as it would seem, with an increasing perversity, to cultivate the *thorns and thistles*, leaving the *good seed* as not worth utilizing.

Amongst the things denominated 'thorns and thistles' we are not to be understood as intending to include the modern methods of mathematical analysis. It is true that in the compound trigonometrical process (under which general expression it is intended to include collectively all of the several forms of applied quantitive mathematics) we have an instrument of great value, a mechanism which is correctly adapted to perform its work with certainty and accuracy, and which is, when properly used, perfectly reliable and trustworthy. It is also true that this mechanism has performed a part in the work of civilization, the value and importance of which, in a human sense, it might be difficult to overstate or to estimate too highly; and it is again also true that the lives of very many able men belonging to successive generations have been employed in constructing, elaborating and perfecting this mechanism, and that it has been now brought, comparatively speaking, to a state of completeness and, almost, of perfection. But all this being admitted and fully appreciated,—it is no less true that this mechanism is only a human contrivance,—it is one form of application only out of a number in which the *material* may be applied; it is the utilization only of a very few facts where the number, from which selection may be

made, is almost unlimited. That the mechanism is of a complicated and elaborate character requiring in its higher forms a special training, a long course of study, and much practice to apply it efficiently and with safety to the purpose for which it is adapted, is undeniable. Is this a recommendation or the reverse? No doubt its application and use had much better be left to those who have had the necessary training and are therefore able to apply it in the proper manner: but is it to the advantage of the community that such potentiality should be confined to a very few individuals? Is there any obvious natural necessity that it should be so? Who shall say that the few facts now utilized are the best adapted for the purpose of such an instrument, or that the form of mechanism now in use is the most simple and objective that can be devised. The evil of separating one division of knowledge from the rest as an exclusive department is manifold and unquestionable, and that it is so would probably be admitted in theory by some, perhaps be readily admitted by many, who are, nevertheless, quite prepared to say to those of their fellow labourers in the field of science who approach too nearly their department.—'This is holy ground—go back: thou art unfit.' Is it, or is it not, true that the language of mathematics is fast becoming an unknown tongue to ordinarily educated men, and that those to whom it is known can scarcely hold converse with their fellows (on any scientific subject) in ordinary language without a feeling of condescension and scarcely without a feeling of impropriety? No man's knowledge is perfect, or nearly perfect, in respect even to one kind or variety of knowledge, and much of the value of any one kind unquestionably consists in its belonging to general science, as the part belongs to the whole. Is it true that the mathematician does now, *in some degree*, regard his fellow-worker who is unpracticed in the calculus and non-conversant with differential-methods as but little better than a publican and heathen?

If it be true that such a result does manifest itself in any considerable degree, it may be pronounced decidedly unwholesome and bad—bad for science and bad for civilization—because mathematical knowledge is a necessity to science and a necessity to civilization.

We have now put before the public demonstration of the problem which determines the quantitive ratio of the perimeter of the circle to the diameter. We know that examination will show the demonstration to be mathematically incontestable, and we know, moreover, that the requisite examination cannot be now much longer delayed. We are mindful, however, that there has been for a long time past a trigonometrical (supposed) solution of the problem, which is looked upon by many persons, educated in this department of science, as a mathematical determination of the question; so that when those persons are asked to examine and consider a geometrical solution of the problem, they reply by requiring to have that which they consider to be adverse demonstration disproved before they are willing to pay any attention at all to the alleged geometrical demonstration. In the circumstances of this particular case there is, at least to some extent, a justification of such a requisition as reasonable, because mathematicians have been exposed to much and continual annoyance on this subject from persons who, having neglected the legitimate means of qualifying themselves to investigate such a subject, have not hesitated to give trouble to and waste the time of other people without first taking the trouble themselves to consider whether their qualifications were such as to entitle them to occupy the attention of others, or to justify them in endeavouring to do so. We will now, therefore, attend to this requisition.

There are several methods in which the trigonometrical process is applied, from each and all of which the same result (supposed solution) is obtained. But these several methods belong essentially to one and the same

process, of which they are merely variations. That process consists in dividing and subdividing the arc of a circle, and in measuring by trigonometry the extremely small portion of the arc, which would result from a long continued process of red ıction by definite division. The application of the process being fundamentally based on an assumption or assumed axiom—that, if the arc of a circle be continually reduced by continued bi-section, the last remaining fraction of the arc is equivalent to the fraction of a straight line. The truth of this fundamental assumption we deny. In regard to the trigonometrical result, our objection is to the application of that result and to the inference that a correct solution of the question is thereby arrived at. The most satisfactory way to specify our objection and to point out the precise locality of the mistake, will be to take Legendre's method of applying the process by inscribing and circumscribing polygons in and about the circle; because Legendre's formal exhibition of the process so applied is generally considered by mathematicians as particularly conclusive and reliable.

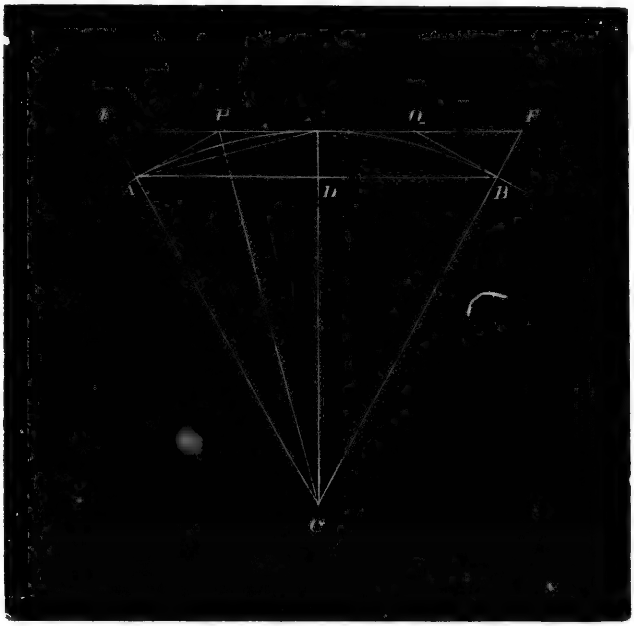
LEGENDRE'S GEOMETRY, (translated by D. Brewster). Book V.—Prop. XIII. Problem.

" *The surface of a regular inscribed polygon and that of a similar polygon circumscribed, being given; to find the surface of the regular inscribed and circumscribed polygons having double the number of sides.*

Let *A.B.* be a side of the given inscribed polygon; *E.F.*, parallel to *A.B.*, a side of the circumscribed polygon; *C.*, the centre of the circle. If the chord *A.M.*, and the tangents *A.P.*, *B.Q.*, be drawn, *A.M.* will be a side of the inscribed polygon, having twice the number of sides; and *A.P.* + *P.M.* = 2 *P.M.* or *P.Q.* will be a side of the similar circumscribed polygon (Prop. VI. Cor. 3.) Now, as the same construction will take place at each of the angles equal to *A.C.M.*, it will be sufficient to consider *A.C.M.* by itself, the triangles connected with it being evidently to each other as the whole polygons of which they form part. Let *A.*, then, be the surface of the inscribed polygon whose side is *A.B.*, *B.* that of the similar circumscribed polygon; *A'.* the surface of the polygon whose side is *A.M.*, *B'.* that of the similar circumscribed polygon: *A.* and *B.* are given, we have to find A'. and *B'.*

First: The triangles *A.C.D.*, *A.C.M.*, having the common vertex *A.*, are to each other as their bases *C.D.*, *C.M.*; they are likewise to each other as the polygons *A.* and *A'.* of which they form part: hence *A* : *A'* : : *C.D.* : *C.M.* Again, the triangles *C.A.M.*, *C.M.E.*, having the common vertex *M.*, are to each other as their bases *C.A.* *C.E.*; they are likewise to each other as the polygons *A.* and *A'*, of which they form part; hence *A.* : *A.'* : : *C.D* : *C.M.* Again, the triangles *C.A.M.*, *C.M.E.*, having the common vertex *M.*, are to each other as their bases *C.A.*, *C.E.*; they are likewise to each other as the polygons *A'.* and *B.* of which they form part; hence *A.'*: *B.* : : *C.A.* : *C.E.*

But since $A.D.$ and $M.E.$ are parallel, we have $C.D. : C.M. :: C.A. : C.E.$,—hence $A : A.' :: A.' : B.$—hence the polygon $A'.$, one of those required, is a mean proportional between the two given polygons $A.$ and $B.$, and consequently $A'. = \sqrt{A. \times B.}$



Secondly.—The altitude $C.M.$ being common, the triangle $C.P.M.$ is to the triangle $C.P.E.$ as $P.M.$ is to $P.E.$; but since $C.P.$ bisects the angle $M.C.E.$ we have $P.M. : P.E. :: C.M. : C.E.$ (Book IV., Prop. XVII.) $:: C.D. : C.A. :: A. : A'.$—hence $C.P.M. : C.P.E. :: A. : A'.$—and consequently $C.P.M. : C.P.M + C.P.E.$ or $C.M.E. :: A. : A. + A'.$ But $C.M.P.A.$, or 2 $C.M.P.$ and $C.M.E.$ are to each other as the polygons $B'.$ and $B.$ of which they form part; hence $B' : B. :: 2\ A. : A. + A'.$ Now $A'.$ has been already determined; this new proportion will serve for determining $B'.$ and give us $B'. = \dfrac{2\ A.B.}{A. + A'.} =$ and thus by

means of the polygons $A.$ and $B.$ it is easy to find the polygons $A'.$ and $B'.$ which shall have double the number of sides.

Prop. XIV. Problem.

To find the approximate ratio of the circumference to the diameter.

Let the radius of the circle be 1; the side of the inscribed square will be $\sqrt{2}$ (Prop. III. Sch.,) that of the circumscribed square will be equal to the diameter 2; hence the surface of the inscribed square is 2, and that of the circumscribed square is 4. Let us therefore put $A.=2$ and $B.=4$; by the last proposition we shall find the inscribed octagon $A'=\sqrt{8}=2\cdot8284271$, and the circumscribed octagon $B.'=\frac{16}{2+\sqrt{8}}=3\cdot3137085$. The inscribed and the circumscribed octagons being thus determined, we shall easily, by means of them, determine the polygons having twice the number of sides. We have only in this case to put $A.=2\cdot8284271$. $B.=3\cdot3137085$; we shall find $A'=\sqrt{A.B.}=3\cdot0614674$, and $B'.=\frac{2\ A.B.}{A.+A'.}$ $=3\cdot1825979$. These polygons of 16 sides will in their turn enable us to find the polygons of 32—and the process may be continued till there remains no longer any difference between the inscribed and the circumscribed polygon, at least so far as that place of decimals where the computation stops, and so far as the seventh place, in this example. Being arrived at this point, we shall infer that the last result expresses the area of the circle, which, since it must always lie between the inscribed and circumscribed polygon, and since those polygons agree as far as a certain place of decimals, must also agree with both as far as the same place.

We have subjoined the computation of those polygons, carried on till they agree as far as the seventh place of decimals:

Number of sides.	*Inscribed polygon.*	*Circumscribed polygon.*
4	2·0000000	4·0000000
8	2·8284271	3·3137085
16	3·0614674	3·1825979
32	3·1214451	3·1517249
64	3·1365485	3·1441184
128	3·1403311	3·1422236
256	3·1412772	3·1417504
512	3·1415138	3·1416321
1024	3·1415729	3·1416025
2048	3·1415877	3·1415951
4096	3·1415914	3·1415933
8192	3·1415923	3·1415928
16384	3·1415925	3·1415927
32768	3·1415926	3·1415926

The area of the circle, we infer, therefore, is equal to 3·1415926. Some doubt may exist perhaps about the last decimal figure, owing to errors proceeding from the parts omitted; but the calculation has been carried on with an additional figure, that the final result here given might be absolutely correct even to the last decimal place.

Since the area of the circle is equal to half the circumference multiplied by the radius, the half circumference must be 3·1415926, when the radius is 1; or the whole circumference must be 3·1415926, when the diameter is 1; hence the ratio of the circumference to the diameter, formally expressed by π, is equal to 3·1415925. The number 3·1416 is the one generally used."

Of these two propositions the last is a computation *

* All computations really such (*i. e.* which are not merely explanatory or merely the numerical equivalents of statements), may be considered the solutions of propositions in the science of quantity and number. If an algebraical computation, the proposition is quantitive; if belonging to arithmetic, the proposition is numerical.

based upon the first, or it may be considered as the numerical equivalent and illustration of the first, exhibiting in numerical units of the radius those relations of the parts which the first proposition apparently establishes. The first of these two propositions (prop. XIII.) is, however, quantitive, and may be also correctly considered as belonging to the science of Quantity and Number * rather than to that of 'Magnitude and Form.' Our objection to these propositions is two-fold, by which we mean that we object to two assumptions involved in these propositions, and which two assumptions, although nearly related, may be considered distinct.

Almost at the commencement of proposition XIII., we find the first assumption, to which we object, distinctly stated in these words: "If the chord *A. M.* and the tangents *A. P.*, *B. Q.* be drawn, *A. M.* will be a side of the inscribed polygon, having twice the number of sides; and *A. P.* + *P. M.* = 2 *P. M.* or *P. Q.* will be a side of the similar circumscribed polygon." Here then we have the statement that *A. P.* + *P. M.* = *P. Q.* To justify the acceptance of this statement it should either be supported by demonstration or be in itself manifestly true. (1) Is it supported by demonstration? We refer as directed to Cor. 3 of Prop. VI., which reads thus: "It is plain that *N. H.* + *H. T.* = *H. T.* + *T. G.* = *H. G.* one of the equal sides of the polygon," &c. Now these lines are in the same case as *A. P.* + *P. M.* = *P. Q.*; so that we find therein not demonstration, but an assertion that the thing stated is a manifest fact. (2) Is it manifestly true? The case in the Corollary, just quoted

* It is not meant that these propositions of Legendre are, for this reason, less reliable or of less value, but the distinction is noted as a protest against calling things which belong to different divisions of science, and which are not the same, by the same name. According to the title of the book, they purport to be propositions in Geometry.

from, is that *N. H.* + *H. T.*, which is an angle and indirectly defined, by the preceding and following propositions, to be a complete angular figure * is equivalent to *H. G.* which is a straight line. Similarly in proposition XIII., the angle *A. P.* + *P. M.* is assumed to equal the straight line *P. Q.*, because either *A. P.* or *P. M.* measured separately as a straight line is apparently equal to one-half the straight line *P. Q.* It is, therefore, assumed as manifest that the vertex of the angle has no quantitive value in itself independently of its sides considered as straight lines.

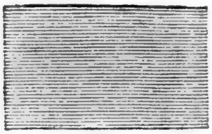


Fig. 1.

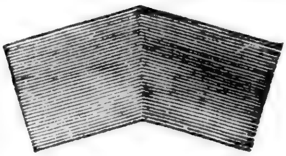


Fig. 2.

Let us suppose Fig. 1 to be a figure compounded of straight lines placed close together, and Fig. 2 to be a figure likewise compounded of angle-lines placed close together. In Fig. 1, if two of the lines be considered relatively to each other, they appear to be similar and equal magnitudes, and on consideration of the figure,—that is, of all the lines compounding the figure relatively to each other,—it becomes manifest that the lines are actually similar and equal longitudinal magnitudes, because both the sides of the figure compounded by the lines are perpendicular to the longitudinal extension of those straight lines, and the compounded figure is rectangular: therefore, if two or more of the lines be similarly divided, the divisional part or parts of the one will be equal to the

* That is—not a compound fragmentary figure formed by two lines merely placed together.

similar divisional part or parts of the others. In Fig. 2, if the two top or bottom angle lines be considered relatively to each other they appear to be also similar and equal magnitudes, but on consideration of the figure, that is, of all the lines compounding the figure relatively to each other, it becomes manifest that the lines, although similar, are not equal, for the lower lines of the figure are evidently lesser longitudinal magnitudes than the upper lines; nevertheless, if two of the angles compared together be considered as compounded of four straight lines, and the four straight lines be compared with each other as longitudinal magnitudes, the difference may be less than any assignable quantity; moreover, if the equal lengths of the lines be equally increased, the extremely minute difference must be proportionately diminished, and, if the equal longitudinal magnitudes be in such wise indefinitely increased, the difference will be indefinitely diminished; nevertheless, the diff. is actual, and if one of the angle lines compounding the figure, however extremely minute may be the breadth which that line is imagined to possess, be (supposed) separated into two lines each having half that breadth, then *must necessarily* the outer of the two halves be of greater length than the inner; and, moreover, the under surface of the outer half must necessarily be in longitudinal magnitude greater than the upper surface of the inner half. Before considering the (1) objection further, we will go on to define the character of the (2) objection which, as we have stated, is in some measure distinct, and we will then consider the two-fold objection as one. In Fig. 3, we have the arc *A.B.*, the sine *C. B.* and the tangent *A. d.* of the arc. Now, there is an assumption, which we are about to explain and to which we are about to object, in rexpect to the diminution of this figure by definite division of the arc, which is at the present time generally adopted by mathematicians; and which, although not

distinctly put forward as an axiom or theorem in either of the two propositions, is adopted also by Legendre, and forms an essential part of the foundation* upon which his apparent demonstration rests. Let us suppose the arc *A. B.* to be bisected, and the sine of the remaining half-arc to be drawn, and the tangent *A. d.* to be also bisected; we shall then find that both the sine and tangent in the lesser resulting figure are much nearer to the arc throughout their length

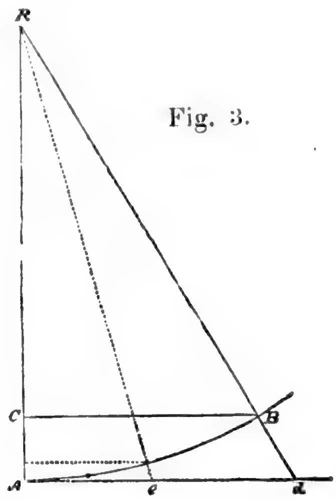


Fig. 3.

than in Fig. 3. If the arc and the tangent of the lesser figure be also bisected, a still nearer approach of the sine, the arc, and the tangent to each other will result, and it is evident that if this process of bisection were to be con-

* The coincidence of the lines appears to be included in, and to be partly the subject of his demonstration; but careful consideration will show that the process itself, having regard to its intended application, is primarily based upon a foregone conclusion as to such coincidence.

tinued in like manner, the deviation of either of the remaining parts of the three lines from a single straight line would be but very small; for, at the 8th bisection, only the 256th part of the tangent *A. d.* would remain, and this would evidently very nearly coincide with the remaining 256th part of the arc. Now, the assumption which we wish to specify, and to which we object, is that—if the process of bisection be so continued, a small part of each of the three lines will eventually remain, which will absolutely coincide and become essentially one line. Observation of Fig. 3 shows that the two extremities of the figure *A. C. B. d.* are not similar; the relations of the three lines—the sine, the arc, and the tangent—are such that if the line *C. B.* be moved down and applied upon the line *A. d.*, a small part of the three lines at the extremity *A.* will appear to coincide; but at the opposite extremity *B.d.*, although the two lines *C.B.* and *A.d.* would in that case coincide, the arc *A.B.* would deviate by the amount of its curvature from the straight line. Now if the usual method of considering the result of the process is reversed, and, instead of considering the eventual relationship of the lines (when the process has been carried very nearly to the vanishing point) with reference to the extremity *A.* and the perpendicular *R. A.*, that eventual relationship is considered with reference to the opposite extremity of the remaining arc and to the radius which intercepts that opposite extremity—the impossibility of an absolute coincidence of the three lines becomes at once apparent, because—however minute the fraction of the arc remaining—*so long as* there be *any arc*, the radius which intercepts the extremity of the arc opposite from *A.* can never become quite perpendicular, consequently the sine must necessarily remain, until the very last, inside the arc, and the tangent must remain outside. The obvious impossi-

bility of an actual coincidence of the three lines would seem to have forced itself for the moment (to be again immediately lost sight of) on the attention of Legendre, and it is admitted in Prop. XIV. by the words "We shall infer that the last result expresses the area of the circle, which, *since it must always lie between the inscribed and circumscribed polygon,* * and since these polygons agree as far as a certain place of decimals, must also agree with both as far as the same place." Now the cumulative character of any disagreement has here been overlooked; a very close approximation to equality in the three lines, left remaining as the result of continued bisection carried to the extreme, is quite intelligible and indisputable, but a very close approximation between the inscribed polygon, the circumscribed polygon, and the arc between them, if that arc contains the eighth part of a circle, becomes, when the construction of the circle is correctly understood, a manifest impossibility. If the increase of the small remaining part of the arc was merely an extension thereof into a greater magnitude of similar form, such as would result from increasing the length of the radius namely, the production of a larger arc similar in form to the small arc, there would then be a possibility of an almost absolute agreement, such as alleged; but a circle cannot be produced in such a way; the circle may be considered as formed by the longitudinal union of extremely small arc magnitudes similar in form to each other; consequently, as the compounded magnitude increases, so does the deviation from the straight line, and so, also, does the

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* We have put these words in italics to specify the admission. It is to be noted that this admission is quite irreconcilable with the alleged agreement or equality of the lines, because such agreement would be equivalent to absolute coincidence of the lines.

amount of difference between the arc and straight line, continually accumulate. The difference which expresses the disagreement in actual length of the small lines, is a part of the *relation in form* of those lines, which increases proportionally to the increasing development and magnitude of the figure. If the small part remaining as the result of bisection be the 100th part of the half-quadrant * then the difference between the sum of the one hundred small sines, the sum of the one hundred small tangents, and the half-quadrant itself will be one hundred times greater than the corresponding difference existent in the small arc between the similar lines; or, if the small arc be the one-thousandth or the one-millionth part of the half-quadrant, then must the difference between the sum of the sines and of the tangents of all the combined small arcs, and the half-quadrant itself be one thousand times, and one million times respectively, greater than the difference between the corresponding lines, to wit—between the sine, the tangent, and the arc-length, belonging to the small arc.

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* The circle naturally divides itself into eight parts compared to an inscribed or circumscribed square, because the four quadrants are each divisible into two half-quadrants, each of which has the same relative relation to the side of the square as the relation of the other half; so that the eight parts of the circle relatively to the sides of the square are similar and equal each to each, but if further division be made, then such equality and similarity in form relatively to the side of the square wiil be no longer obtained, because the curvature of the diminished arc will be less than that of the part divided off from it. The non-appreciation of this fact has much to do with the erroneous conclusion supposed to have been demonstrated; it is, indeed, in the distinct and thorough apprehension of this relationship that an explanation of the precise character of the fallacy is to be found.

Some of those persons who are reasonable enough to feel that to nominally define a thing by calling it an abstraction does not define it at all, prefer to adopt the expression 'surface' whereby to denote a line, * and define a geometrical line as 'the surface of a geometrical figure.' Now the surface of a thing must either belong to and be a part of the thing or must be outside of it . . . in regard to the circle, for instance, the surface must either belong to and be a part of the area of the circle or it must be outside the circle. If supposed to be within the circle then must the circumscribed polygon, compounded by the union of the small tangents, (if the circumscribed

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* Strictly speaking, according to the presently accepted doctrine, a line is the extremity of a superficies. If a superficies were to be defined as a real surface, compounded of real lines, the expression would not, we opine, be subject to objection ; a line might be then considered as one of the extremities of the surface, or as a section of, or as one of the elementary constituents of the surface. As the (so called) definition now stands, if we attempt to directly cognize it as an intelligible idea, we find ourselves almost immediately enveloped in a network of contradictions. (1) The surface is either compounded of lines, or it is not compounded of them.—If it is, how can that which hath breadth be compounded of that which is without breadth? If it is not; what then is that surface of which the line is the extremity, and of which, nevertheless, the line is not a part? of what then is that surface compounded, and is the surface itself a part of, or does it belong to anything? But the difficulty (dilemma) is of a still more refined character.—"The extremities of a line are points," which are negatively *defined* to be *nothing*. Now, the relation of the line to the surface is defined to be similar to the relation of the point to the line: it therefore seems to follow that since the line has nothing for its extremities, the surface likewise has nothing for its extremities. How are we to cognize the idea of a surface without any extremities; or, of a surface which hath length and breadth together with extremities which, according to the definition, certainly have no breadth and probably have no length, for the supposition of a line without extremities possessing length is, if it have any meaning, the negative supposition of a non-existent line which might be possessed of length if it were existent.

(See, in the Appendix, Dr. Simson's explanation of a superficies, and illustration of Euclid's dogma.)

polygon be a continuous and not a fractional figure) be necessarily greater than the surface. And if the surface be outside the circle, then must the surface be greater than the sine, which is always within the circle and therefore less than the circumscribing polygon by which the circle is surrounded.

The objections to Legendre's propositions, as a supposed demonstration of the ratio of the circle to the diameter, are:—

First.—Evidently, he assumes the circumscribed polygon to be a simple continuous figure; because the application of his propositions is based upon a comparison of the polygon with the arc of the circle which is a simple continuous figure, and if the polygon be not simple and continuous his demonstration fails, since, in that case, the reasonableness of the application is not shown. It is objected that the case is not in fact as he assumes it to be: the circumscribed polygon is not simple and continuous, but compound and fragmentary . . for, if a straight line be bisected and the two halves thereof be placed together, so that their adjoining extremities form the vertex of an angle, it is manifest that the quantity of length contained in the exterior surface of that angle must be, measured as a continuous surface, greater than the sum of the two lines, measured separately and taken together; because the interior surface (within the angle) must be equal to the lengths of the two lines taken together, and the exterior surface of an angle, considered as a complete figure, is evidently greater than the interior surface.

Second.—The inscribed polygon is assumed, if the process of bisection be continued, to become eventually coincident with the circle. It is objected that this is impossible, for so long as any arc remains, however minute that part may be, the sine of the arc must be within the arc; and, if this be admitted, (which it must

necessarily be) it then follows that, in comparing the half quadrant with the radius, the difference, because cumulative, must be increased proportionally to the magnitude of the half-quadrant compared with that of the minute arc.

Some mathematicians are disposed to require in this case, in addition to adverse demonstration and to reasonable objection, the explanation of an apparent difficulty of a particular kind. It is said :—there can be no doubt that this numerical quantity found by Legendre and others as the ratio of the circle to the diameter, does represent the ratio of a quantity which has some actual and significant relationship to the circle; that it must be so, is established by indirect evidence, for the very same quantity appears as the notable result of quite a number of different computations connected with the circle. Explain, therefore, what this quantity really is. What is the relationship of this quantity if it be not in fact that assigned to it by Legendre and others?

The answer to this requisition is :—The quantity in question, namely—·78539 . . . . is the sum of the sine-lengths of all the elementary arcs contained in the half-quadrant, and into which the half-quadrant may be divided by the continued process of bisection. In other words, 78539 . . . . represents the length of the sine belonging to the extremely minute (ultimate) arc, which results from the continued process of bisection, multiplied by the number of those minute arcs contained in the half-quadrant, when the radius of the circle is valued as a unit.*

The origin of the error in Legendre's method, as well

* This fact, with its precise significance, may become more distinctly apparent by consideration of the process of the continued duplication of the continually bisected arc—an explanation of which will be found in the Appendix.

as in the application of the process known as that of continued bisection, is in the omission to observe that comparison has to be made between a continuous curved line (the circle) and a continuous straight line (the diameter.) The circle may be divided into four equal parts, and the inscribed (or circumscribed) square is then equally and similarly related to each of those parts; and, further, each of the quadrants and each side of the square may be bisected, and still the same equal and similar relationship will exist between each of the eight parts of the curvilinear figure and each half-side of the square; but, if division be carried further, the relationship will no longer continue similar and equal. If the half side of the square were to be broken up into its (ultimate) component parts, and these arranged as an inscribed (or circumscribed) polygon, each minute part would be related to its arc—similarly to each of the other minute parts; but the polygon, being fragmentary and non-continuous, would not be in the same case as, and would not admit of indiscriminate comparision with, a continuous straight line.

For the purpose of defining the characteristic distinction between a straight line and an arc of a circle, and, at the same time, indicating the boundary between the conceivable and the inconceivable—between the reality of Science and unreality of Metaphysics,—we will notice here the word 'infinite.' In the first place it may be of service to call attention to the foolish and mischievous manner in which this word is becoming more and more frequently used. The word 'infinite' is properly and correctly used in Mathematics and in other divisions of science as the opposite to 'finite.' Outside or independently of its use in such sense the word 'infinite' has a naturally sacred character as applying to the attributes of the Creator, and it is very desirable that the use of the word should be restricted to its proper and correct

application. It is now used in all sorts of literary composition,—in newspaper articles, lectures, sermons, &c.—in a manner that may be termed *unsense*, and which would oftentimes be absurd if it were not calculated to have a seriously mischievous effect. Perhaps, the most common mistake is its use as equivalent to the expressions—'indefinite,' which means that which is finite, but of which the limit is not, or cannot be, defined—and 'immeasurable,' which means that which cannot be measured in consequence of its exceeding greatness.

'Infinite' means boundless, unlimited, endless, continual extension, &c., &c. The word does not correctly compound with words (adjectives) which express comparative extent, or in which measurement is implied. Such a compound expression as 'infinitely great' is, therefore, (with exception of the theological sense) barbarous; 'infinitely greater' or 'wiser' is about equally so; 'infinitely small' and 'infinitely less,' or 'infinitely more foolish' are perhaps even worse. We make these remarks in this place, primarily, for the purpose of obtaining the requisite attention to the distinction between a circle described (1) with a definite radius or (2) with a radius of indefinitely great length, and (3) a circle described with (using the expression for illustration only) an infinite radius. In the first there is included, because necessarily resulting therefrom, the idea of a definite circle, of which the magnitude is determined and defined by the definite magnitude of the radius. Belonging to the second is a circle the magnitude of which is limited; its magnitude may be immensely, perhaps immeasurably, great, and of unknown greatness, but it is limited by the limited length of the radius, whatever that length may be . . for example, the radius, or radial distance, may be the distance between the earth and the most distant star, and, as the distance of the star is evidently finite, so must the magnitude of the circle described with that

radial distance be limited accordingly. It may be here noted that we can conceive or adopt as a conception and distinctly cognize as an idea, anything of which we have (obtained) certain knowledge as a fact . . . taking the example just stated, or the similar case—of the earth moving in its orbit of revolution around the sun, if we consider the motion of the earth through a quantity (an extension) of space equivalent to a few yards or feet of terrestrial measurement, the motion would appear to be in a straight line. Compared to any merely terrestrial motion, it would be almost absolutely in a straight line; and, considered merely in reference to any such motions, its deviation from a straight line would be inconceivably small, and would scarcely admit even of intelligible expression as a comparison with any merely terrestrial quantity; nevertheless, by a knowledge of the fact, and of certain other facts with which to make comparison, we are enabled not only to obtain the conception of, but also to distinctly appreciate the deviation and even to determine and measure its amount with almost perfect accuracy; we are enabled to be perfectly sure that the path or line of the earth's motion, throughout that quantity of space, is not in a straight line but in the arc of a circle, having the sine of the arc within and the tangent of the arc without, the boundary (perimeter) of that circle.

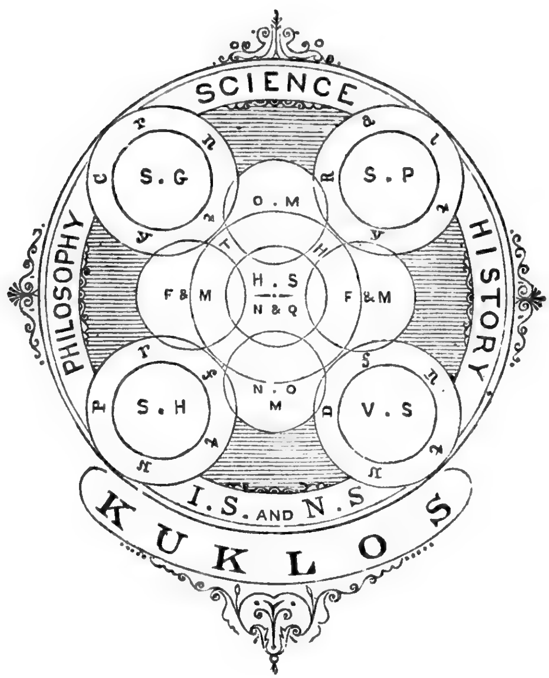
Thirdly, (3) we have the idea of a straight line, or, so to speak, we have the compound idea* of the arc of a circle which has become a straight line. If it were conceivable that a radius-vector might have infinite length, it would then follow that the extremity of such a radius would describe a straight line, and the straight line

* We have already stated that these expressions are used, in this connection, only in illustration of the par' cular case under consideration.

might then be considered as the arc of a circle of infinite magnitude. It may be that herein we have the nearest approach that can be made by the human mind to a conception of that which, if it can exist, does not belong to the world of humanity; to a conception of the inconceivable; for, in what may be termed a compound negative sense, the mind does seem to be very nearly able to distinctly cognize the supposition as a positive idea. Having certain knowledge of the straight line as a fact, and apprehending the necessity that, if the supposition of the infinite radius were admissible, the straight line must have that relation to it which the arc of the circle has to a finite radius—the shadow seems to assume substance, and it is almost as though the mind were able to grasp the unreal idea as a reality. But, in fact, there would be no circle and the supposition as a positive hypothesis does not belong to (human) science. To entertain and to wilfully play with such an unreal idea as a positive or concrete conception is forbidden; to do so would be to leave the realm of science, and, contemning the guidance and authority of reason, to enter the dark domain of metaphysics. We believe, however, it is admissible to entertain the idea, thus far, negatively, for the express purpose of defining the boundary and of clearly realizing the essential distinction between the straight line and the circle.

A very serious obstruction in the way of intellectual progress has been now pointed out and clearly indicated. The false statement at the commencement of Euclid's great work is now unmasked and its actual character made manifest as a monstrous deception of Untruth in the place of that which it purports to be—a true definition of reality. Those who for the future are deceived by it must be so by wilfully subjecting themselves to such deception. So soon, however, as its false and deceptive

character has become distinctly understood, those who more particularly represent Education and Science, should mark its baneful influence in the vantage ground it has so long occupied, and see to its speedy condemnation and removal. Let 'the stumbling-block be taken up out of the way.'



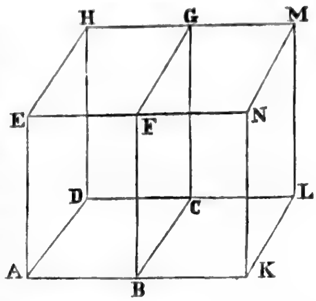
APPENDIX.

An ingenious endeavour has been made by Dr. Simson, (and others,) to give an intelligible explanation of the compound dogma prefixed to Euclid's work, by means of the figure of a solid cube or parallelopiped. The solid is supposed to be evenly divided by a section at right angles to its sides. It is then explained (argued) that the section is the superficies, and that the superficies cannot belong to either part of the solid, because if either one of the parts be removed the surface still remains with the other part; and hence it is inferred that the surface has no breadth; &c., &c., as follows:

Dr. Simson's Explanation.—" It is necessary to consider a solid, that is, a magnitude which has length, breadth and thickness, in order to understand aright the definitions of a point, line, and superficies; for these all arise from a solid, and exist in it. The boundary, or boundaries, which contain a solid, are called superficies, or the boundary which is common to two solids which are contiguous, or which divides one solid into two contiguous parts, is called a superficies: thus, if BCGF. be one of the boundaries which contain the solid ABCDEFGH., or which is the common boundary of this solid and the solid BKLCFNMG., and is therefore in the one as well as the other solid, it is called a superficies, and has no thickness; for if it have any, this thickness must either be a part of the thickness of the solid AG., or the solid BM., or a part of the thickness of each of them. It cannot be a part of the thickness of the solid BM.; because, if this solid be removed from the solid AG., the superficies BCGF., the boundary of the solid AG., remains still the same as it was. Nor can it be a part of the thickness of the solid AG., because if this be removed from the solid BM., the

superficies BCGL. the boundary of the solid BM., does nevertheless remain; therefore the superficies BCGF. has no thickness, but only length and breadth.

"The boundary of a superficies is called a line; or a line is the common boundary of two superficies that are contiguous, or it is that which divides one superficies into two contiguous parts: thus, if BC. be one of the boundaries which contain the superficies AB CD., or which is the common boundary of this superficies and of the superficies KBCL. which is contiguous to it, this boundary BC. is called a line, and has no breadth; for, if it have any, this must be part either of the breadth of the superficies ABCD. or of the superficies KBCL., or part of each of them. It is not part of the breadth of the superficies KBCL.; for, if this superficies be removed from the superficies ABCD., the line BC., which is the boundary of the superficies ABCD., remains the same as it was. Nor can the breadth that BC. is supposed to have be a part of the breadth of the superficies ABCD.; because, if this be removed from the superficies KBCL., the line BC., which is the boundary of the superficies KBCL., does nevertheless remain: Therefore the line BC. has no breadth; and, because the line BC. is a superficies, and that a superficies has no thickness, as was shewn; therefore a line has neither breadth nor thickness, but only length.



"The boundary of a line is called a point, or a point is a common boundary or extremity of two lines that are contiguous: Thus, if B. be the extremity of the line AB. or the common extremity of the two lines AB., KB., this extremity is called a point, and has no length; for if it have any, this length must either be part of the length of the line AB. or of the line KB. It is not part of the length of KB.; for, if the line KB. be removed from AB. the point B. which is the extremity of the line AB., remains the same as it was; nor is it part of the length of the line AB.; for

if AB. be removed from the line KB., the point B., which is the extremity of the line KB., does nevertheless remain: Therefore the point B. has no length; and, because a point is in a line and a line has neither breadth, nor thickness, therefore a point has no length, breadth nor thickness. And in this manner the definition of a point, line, and superficies are to be understood."

We object to this explanation; that..the boundary defined is a space outside the solid: for (1) the boundary which surrounds and contains a solid must be greater than and must be outside the solid which it contains, and (2) the boundary which separates and is common to the contiguous sides of two solids is a space—which space must contain more or less breadth according to the greater or less distance between the two solids (or two parts of the solid.) Let the distance between them be any definite small quantity of space......the two contiguous solids (or parts of the solid), may be then removed to twice that distance, and the boundary which is common to both will have twice the amount of breadth...... or, the two solids may be brought nearer together and the distance between them lessened to the one-half, the boundary common to both will then have one-half the breadth it previously had, or the distance may be lessened to the one millionth part and then will the breadth of the boundary be diminished to the one millionth part of the breadth contained in the previous boundary, and so on. But now, if the two solids (or the two parts of a solid) are brought into absolute proximity and united into one solid (or into a whole undivided solid,) the boundary common to the two solids and which is outside each of the solids (and is a part of the boundary containing each of the solids respectively) has disappeared, it is no longer between them for they are united: evidently the boundary of the solid (or part) ABCDEFGH., if it be outside that solid must be now a part of the other solid (or part) BKLCFNMG., and inversely the boundary of the last, if it be outside the last solid itself, must be a part of the first.

The reasoning applies equally to the similarly related case of *the line* in which the line BC. must be a breadth-containing space between the two contiguous superficies and be common to both of them; for if they be united, a boundary

in the same sense is no longer existent; neither is it conceivable—for, to conceive *such* a boundary is to include the idea of separation which necessitates the cognition of a space measured by the distance (amount) of that separation. And it is evident that the space is the line so conceived or cognized.

Similarly in regard to the point B. If the point be the extremity of the line AB., and be in the line; then is it a part of the line AB., and if the line AB. be removed, then is the point B. removed with it; for, to assert the contrary is to assert that the same one point B. can at the same time be in two or more different places, which is absurd. And again, if it be neither in the line AB., nor in the line KB., then it cannot be the extremity of either line, in the sense of belonging to and ending the line, therefore it must be the space between the extremities of the lines and must contain magnitude (breadth) limited and measured by the distance which separates those extremities, (or a minute and definite divisional part of that space may be taken to represent the line; i.e., the line may be considered as constituted by the space, or as contained in the space and constituted by a divisional part of the space.)

THE ULTIMATE SINE.

The repeating process of duplicating the bisected arc.—The result of this process is to obtain the sine length belonging to the minute (ultimate) part of the arc which would remain after the process of bisection had been repeated a great number of times—that is, repeated until the vanishing-point had been almost arrived at. It is, therefore, strictly speaking, *a method* belonging to the same general process of which several methods are already known and practised, amongst them being that of Legendre which we have quoted.

The method which we are now about to explain has the advantage, we think, of exhibiting the facts from which the elements of the computation are derived, in a more simple, direct, and readily intelligible form; and perhaps, also, in a more generally instructive and useful form.

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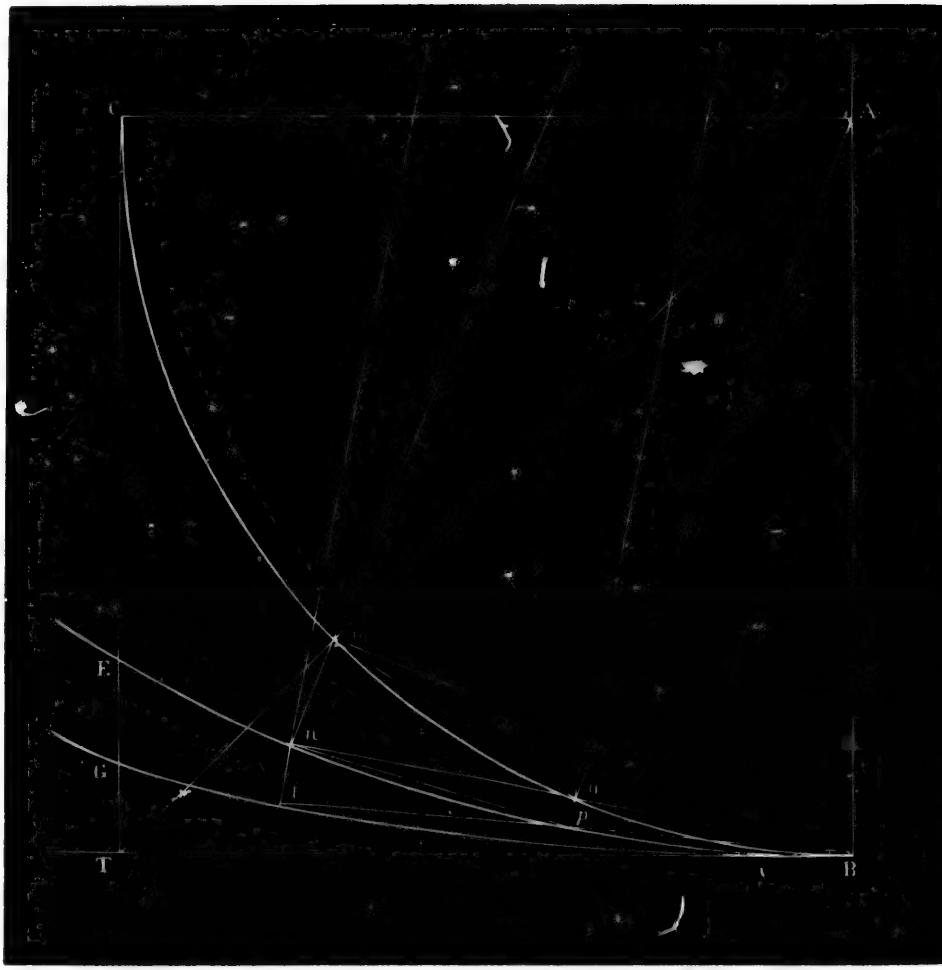
FIG. 18.



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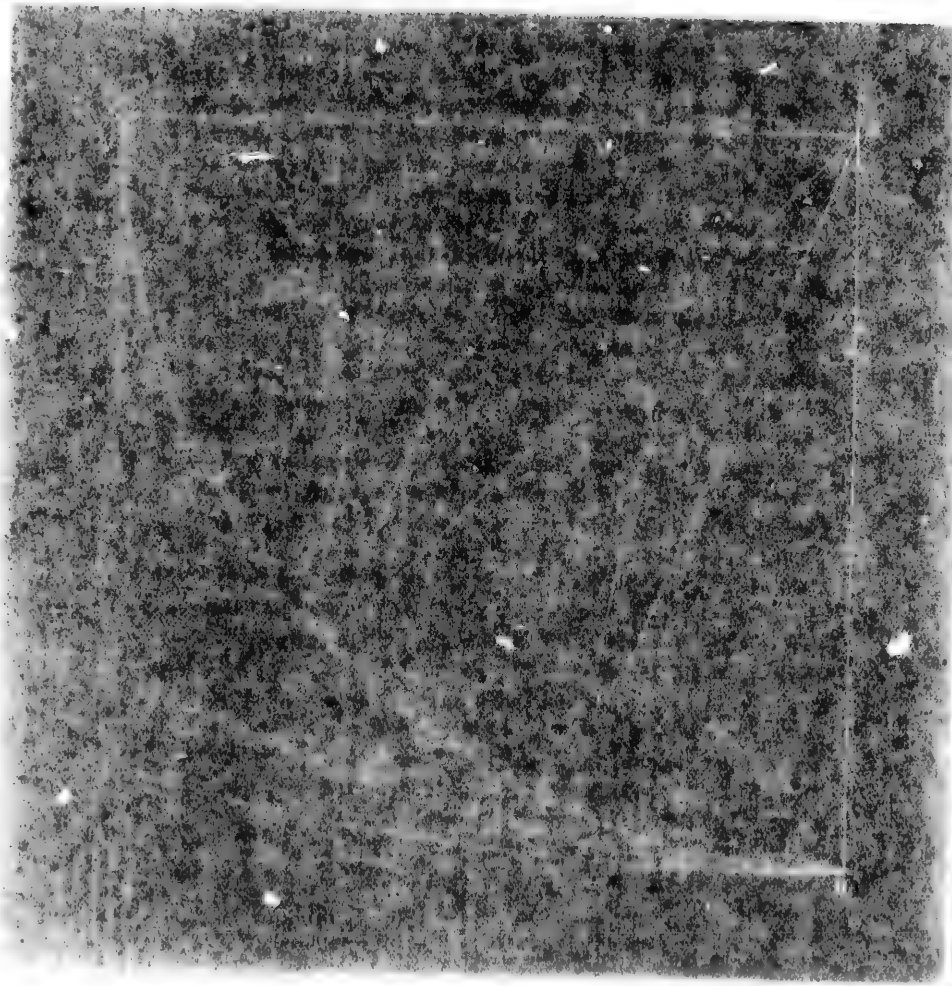

The method may be thus explained:—The half quadrant (or any other definite fraction of a circle) being described with a given radius of definite length—(1) The chord of the arc is drawn; through the terminal point of the arc, and at right angles to the chord, a line is drawn (of indefinite length,) which line is a part of the secant belonging to half of the arc duplicated in magnitude. (2) The arc is bisected and through the point of bisection, from the extremity of the radius, a line is drawn which intercepts the secant of the duplicated half-arc; the line last drawn is therefore the chord of the half-arc duplicated, and the point at which that line intercepts the secant is the terminal point of the half-arc of duplicated magnitude: that is, if the radius of the primary arc were to be duplicated * and a half-quadrant described therewith, the point thus found would necessarily bisect that half-quadrant. (See Fig. 18.) In like manner, a line is drawn through the point thus found at right angles to its chord, which line is a part of the secant belonging to the second half-arc (the half of the duplicated half-arc) duplicated in magnitude. Now in order to draw the chord of the third arc, without actually describing the second arc, the chord, or any part of the chord, of the second arc, may be taken as a radius and an arc described therewith intercepting the tangent of the primary arc; the arc so described being bisected, the point of bisection would be necessarily a point in the line drawn from the extremity of the radius through the point of bisection of the first duplicated half-arc (if the arc be described and the line be drawn.) And the line being drawn from the extremity of the radius through the point thus found and produced until it intercepts the secant last drawn, the point of interception will be the terminal point of the second half-arc quadrupled in magnitude, and the line joining that point with the extremity of the radius is the chord belonging to the second half-arc quadrupled. In like manner the process may be carried on so long as any space remains between the tangent of the primary arc and the terminal extremity of the arc last drawn.

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* Meaning the radius duplicated by doubling the distance of the centre, with which the arc is described, from the point at the original extremity of the arc.

The foregoing is a brief explanation of the general method which, however, admits of several variations by means of which the construction may be made to more completely exhibit and illustrate the application of the method to the purpose of measuring and investigating the lines pertaining to the successive arcs. Fig. 20, for example, in addition to the preceding explanation, which is illustrated therein, shows a convenient method of obtaining the terminal parts of the radii belonging to the successive arcs respectively.

In order to make the reliability and computative value of the method distinctly apparent, we will now, taking the radius equal to 10, and commencing with the half quadrant, carry the computation through eight successive bisections, and then compare the results with those obtained by Legendre from the process of inscribing and circumscribing polygons.

(*Note.* The square root of 'the square of the radius less the square of the sine,' deducted from the radius, gives the versed sine.)

The square of the sine together with the square of the versed sine give the square of the chord.

The chord of (the half quadrant) $\frac{C}{8}$ equals the sine of $\frac{C}{16}$ duplicated; and so on.

Radius = 10.

The half-quadrant R = 10	$\frac{C}{8}$	Sine	$7\cdot071068^2$	=	$50\cdot000000$
		V. S.	$2\cdot92893^2$	=	$8\cdot578631$
		Chord	$7\cdot653667^2$	=	$58\cdot578631$
1st Bisectn. R = 20	$\frac{C}{16}$	Sine	$7\cdot653667^2$	=	$58\cdot578631$
		V. S.	$1\cdot52241^2$	=	$2\cdot317732$
		Chord	$7\cdot803612^2$	=	$60\cdot896363$
2nd Bisectn R = 40	$\frac{C}{32}$	Sine	$7\cdot803612^2$	=	$60\cdot896363$
		V. S.	$\cdot76859^2$	=	$\cdot590745$
		Chord	$7\cdot84137^2$	=	$61\cdot487108$
3rd Bisectn. R = 80	$\frac{C}{64}$	Sine	$7\cdot84137^2$	=	$61\cdot487108$
		V. S.	$\cdot38522^2$	=	$\cdot148394$
		Chord	$7\cdot850827^2$	=	$61\cdot635502$

4th Bisectn. R=160	$\frac{C}{128}$	Sine	$7·850827^2$	=	61·635502
		V. S.	$·19271^2$	=	·037249
		Chord	$7·85319^2$	=	61·672751
5th Bisectn. R=320	$\frac{C}{256}$	Sine	$7·85319^2$	=	61·672751
		V. S.	$·0964^2$	=	·009293
		Chord	$7·85379^2$	=	61·682044
6th Bisectn. R=640	$\frac{C}{512}$	Sine	$7·85379^2$	=	61·682044
		V. S	$·0474^2$	=	·0022457
		Chord	$7·853934^2$	=	61·6842897

The following is in correction of an error in the last part of the table on page 55. Appendix, to Part Second.

6th Bisectn. R=640	$\frac{C}{512}$	Sine	$7·85379^2$	=	61·682044
		V. S	$·0482^2$	=	·0023232
		Chord	$7·85393^2$	=	61·6843672
7th Bisectn. R=1280	$\frac{C}{1024}$	Sine	$7·85393^2$	=	61·6843672
		V. S.	$·0241^2$	=	·0005808
		Chord	$7·85396^2$	=	61·6849480
8th Bisectn. R=2560	$\frac{C}{2048}$	Sine	$7·85396^2$	=	61·6849480
		V. S....................................			
		Chord.......			

128	=	7·850828	7·850827	128th	“
256	=	7·853193	7·85319	256th	“
512	=	7·853785	7·85379	512th	“
1024	=	7·853932	7·85393	1024th	“
2048	=	7·853969	7·85396	2048th	“

It is sufficiently evident that the two methods of computation are precisely equivalent in their results. To correctly appreciate the true relationship and significance of that re-

* The tabulated figures of Legendre's computation will be found at page 33.

The foregoing is a brief explanation of the general method which, however, admits of several variations by means of which the construction may be made to more completely exhibit and illustrate the application of the method to the purpose of measuring and investigating the lines pertaining to the successive arcs. Fig. 20, for example, in addition to the preceding explanation, which is illustrated therein, shows a convenient method of obtaining the terminal parts of the radii belonging to the successive arcs respectively.

In order to make the reliability and computative value of

R = 20	16	Chord	$7 \cdot 803612^2$	=	$60 \cdot 896363$
2nd Bisectn	$\frac{C}{32}$	Sine	$7 \cdot 803612^2$	=	$60 \cdot 896363$
		V. S.	$\cdot 76859^2$	=	$\cdot 590745$
R = 40		Chord	$7 \cdot 84137^2$	=	$61 \cdot 487108$
3rd Bisectn.	$\frac{C}{64}$	Sine	$7 \cdot 84137^2$	=	$61 \cdot 487108$
		V. S.	$\cdot 38522^2$	=	$\cdot 148394$
R = 80		Chord	$7 \cdot 850827^2$	=	$61 \cdot 635502$

4th Bisectn. R=160	$\frac{C}{128}$	Sine	$7·850827^2$	=	61·635502
		V. S.	$·19271^2$	=	·037249
		Chord	$7·85319^2$	=	61·672751
5th Bisectn. R=320	$\frac{C}{256}$	Sine	$7·85319^2$	=	61·672751
		V. S.	$·0964^2$	=	·009293
		Chord	$7·85379^2$	=	61·682044
6th Bisectn. R=640	$\frac{C}{512}$	Sine	$7·85379^2$	=	61·682044
		V. S	$·0474^2$	=	·0022457
		Chord	$7·853934^2$	=	61·6842897
7th Bisectn. R=1280	$\frac{C}{1024}$	Sine	$7·853934^2$	=	61·6842897
		V. S.	$·0224^2$	=	·0005017
		Chord	$7·853966^2$	=	61·6847914
8th Bisectn. R=2560	$\frac{C}{2048}$	Sine	$7·853966^2$	=	61·6847914
		V. S..			
		Chord..			

T· compare Legendre's computation * according to the method of inscribing and circumscribing polygons: Since he takes the whole circle and the diameter as unity, his figures must be multiplied by 10 and divided by 4.

The inscribed polygon of Legendre's method; multiplying the figures by 10, and dividing by 4, to compare with the half quadrant and with R=10.

The bisected are duplicated. The sine lengths belong to the successive arcs of bisection when increased in magnitude to equal the half quadrant.

The number of sides.		The lengths.	The lengths.	The fractional parts of the circle.
8	=	7·071068	7·071067	The 8th part.
16	=	7·653668	7·653667	16th "
32	=	7·803613	7·803612	32nd "
64	=	7·841371	7·841371	64th "
128	=	7·850828	7·850827	128th "
256	=	7·853193	7·85319	256th "
512	=	7·853785	7·85379	512th "
1024	=	7·853932	7·85393	1024th "
2048	=	7·853969	7·85396	2048th "

It is sufficiently evident that the two methods of computation are precisely equivalent in their results. To correctly appreciate the true relationship and significance of that re-

* The tabulated figures of Legendre's computation will be found at page 33.

sult it is necessary to particularly note that the radius of each successive arc is of increased magnitude; in such wise that the seventh arc, resulting from the process, is comparable with its radius which equals 1280; whereas the primary arc is comparable with its radius which equals 10. Therefore, although the seventh arc is absolutely the same length as the primary arc and may be, in that sense, correctly considered as the same arc from which the greater part of its curvature has been eliminated, yet relatively to the radius and therefore relatively to the complete circle (or to the half quadrant) the seventh arc is the 128th part of the primary arc divided off therefrom by repeated bisection and, relatively to the tangential straight line, dissimilar therefrom in form. It now clearly appears that, although the length of the sine belonging to the seventh arc approximates to the length of that arc, since 128 of these fractional arcs must be combined in order to reproduce the half-quadrant, the difference between the sine and arc length of the seventh arc, whatever that difference may be, is subject to multiplication by 128, in order to obtain the difference represented by the ratio of that sine-length when increased by 128 magnitudes, to the arc-length of the half-quadrant.

An advantage of this method for the purpose of quantitive investigation, is that the one continued computation determines the sine lengths belonging to each of the successive arcs. The arcs are all equal in length each to each, and as the curvature is eliminated from the arc by the successive bisections and duplications of magnitude, the ratio of the sine length to the arc-length of the arc continually approaches equality, (i.e. until the ultimate limit at the vanishing point of the circle is reached.) Much facility in carrying on the computation arises from the fact, which an examination of the figure will make apparent, that the chord of the one arc is the sine of the arc next succeeding; for example, the chord of the primary arc of 45 degrees is the sine of the duplicated arc of $22\frac{1}{2}$ degrees. The chord of the arc of $22\frac{1}{2}$ degrees is the sine of the duplicated arc of $11\frac{1}{4}$ degrees, and so on. It is to be also observed that the square of the chord length, by means of which the quantitive value of the versed sine to each successive arc is obtained, is continually furnished by the computation itself.

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FIG.1

FIG.10.

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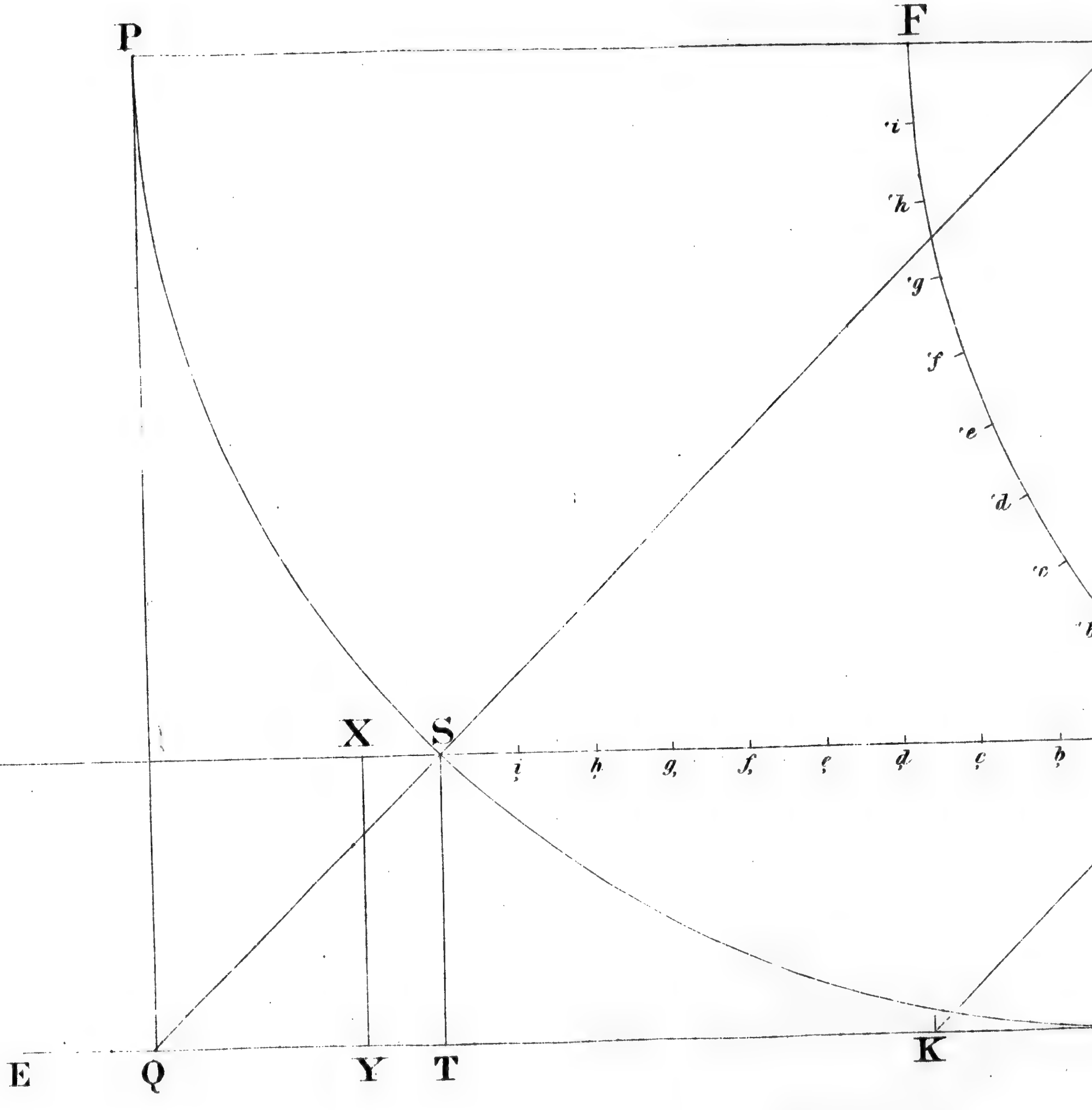


FIG.10.

FIG.11.

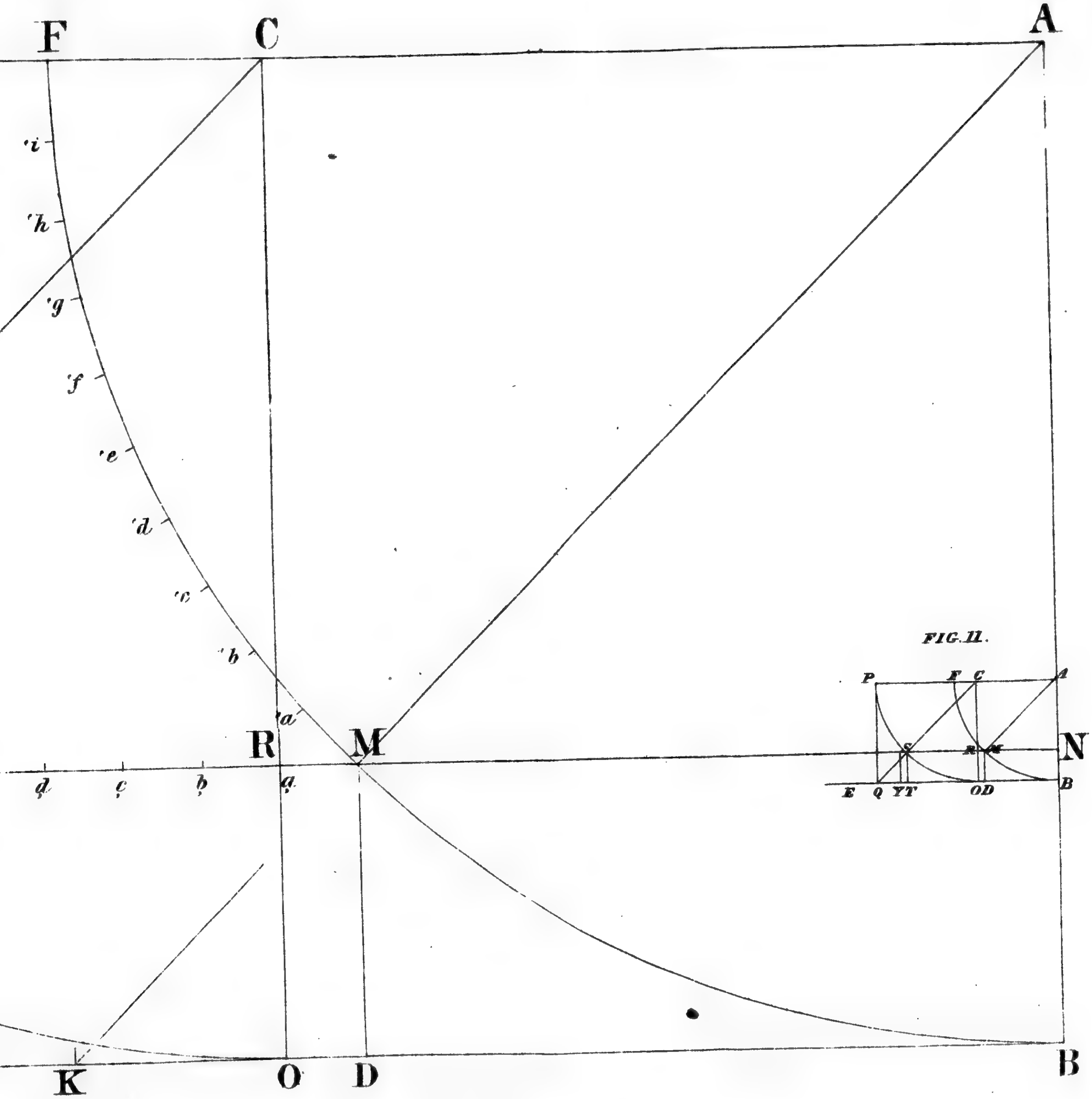


FIG. 12. (a)

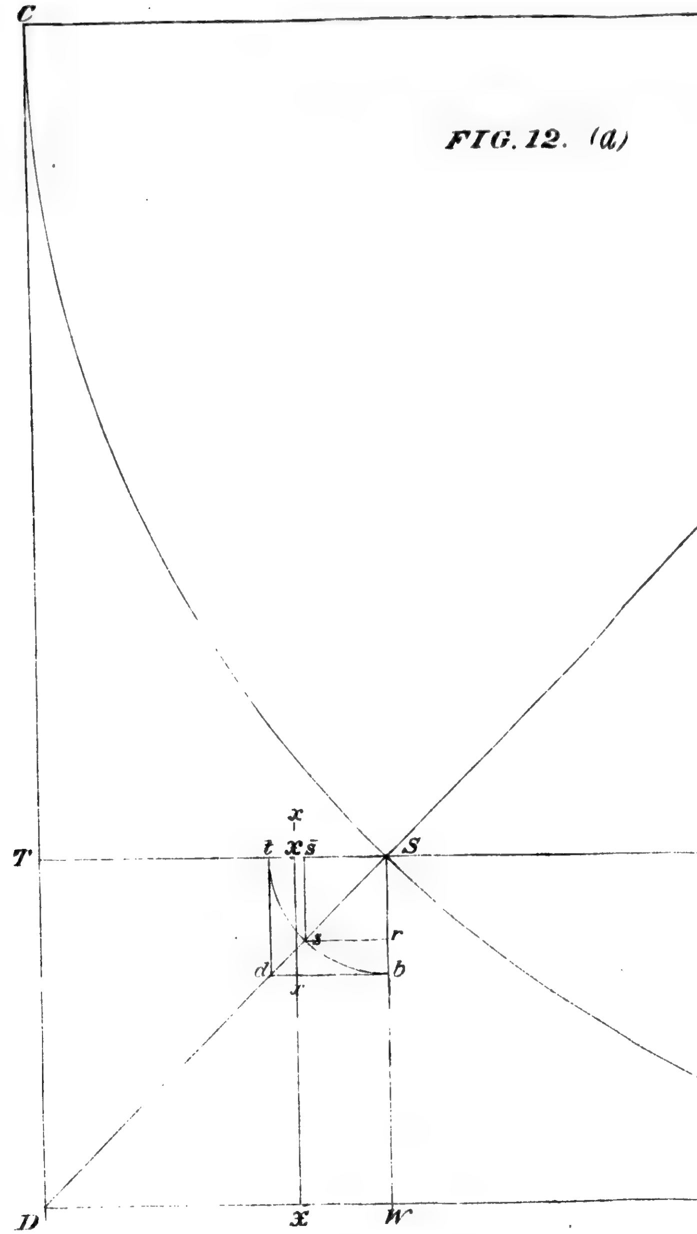


FIG. 12. (a)

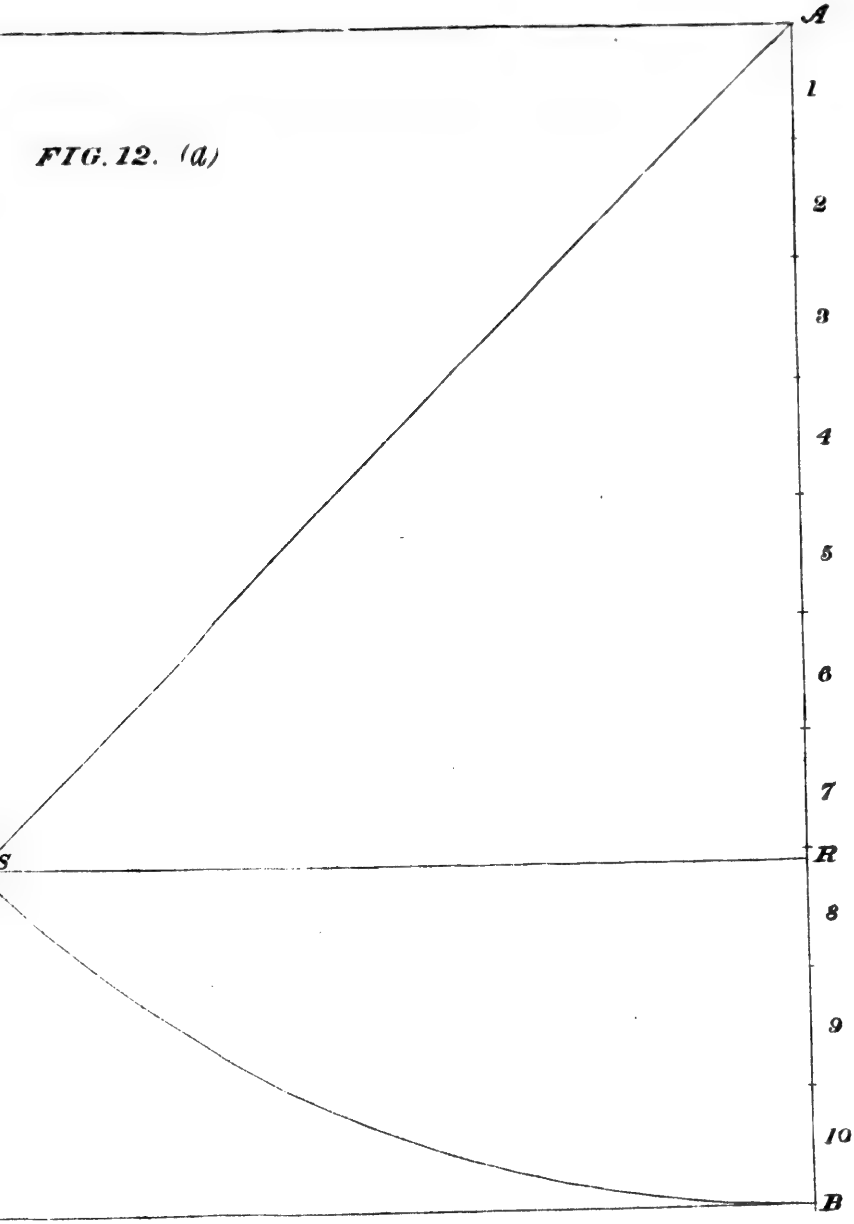
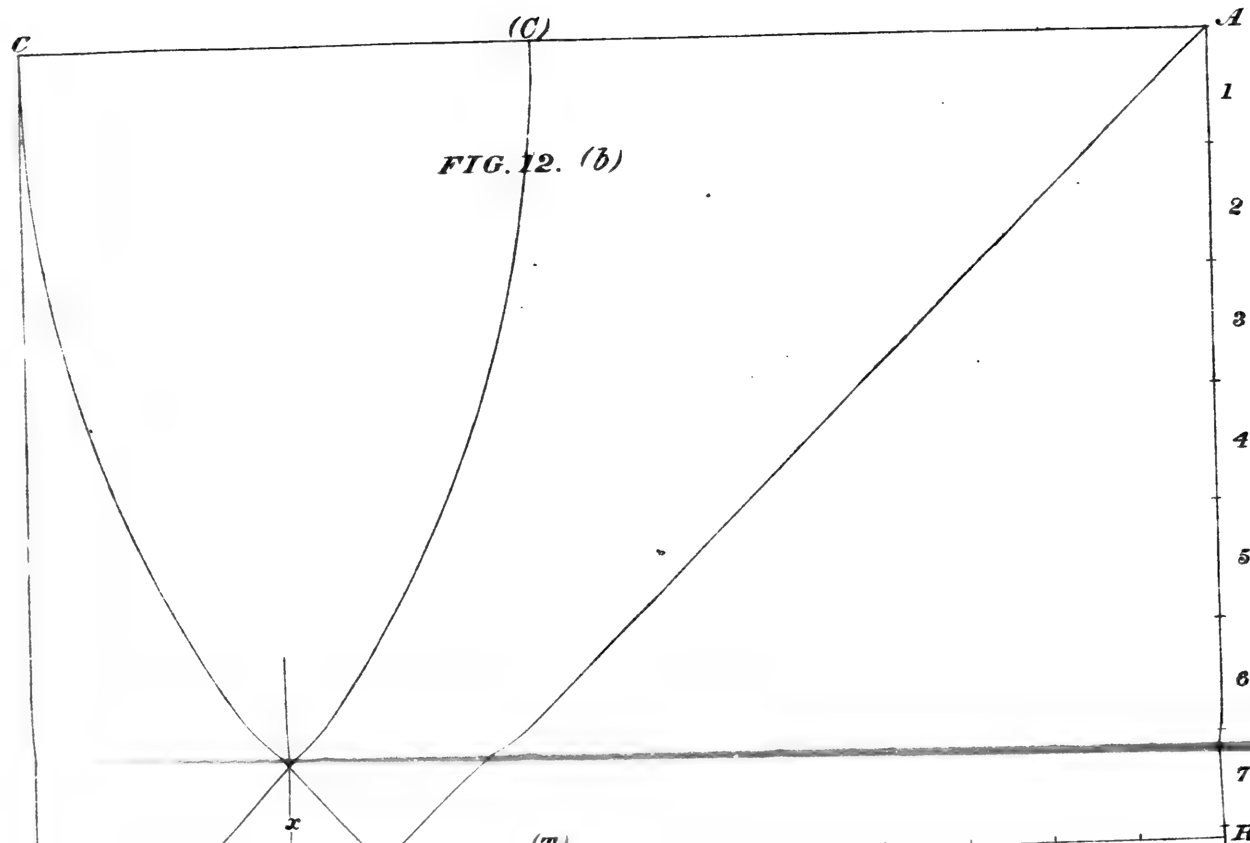
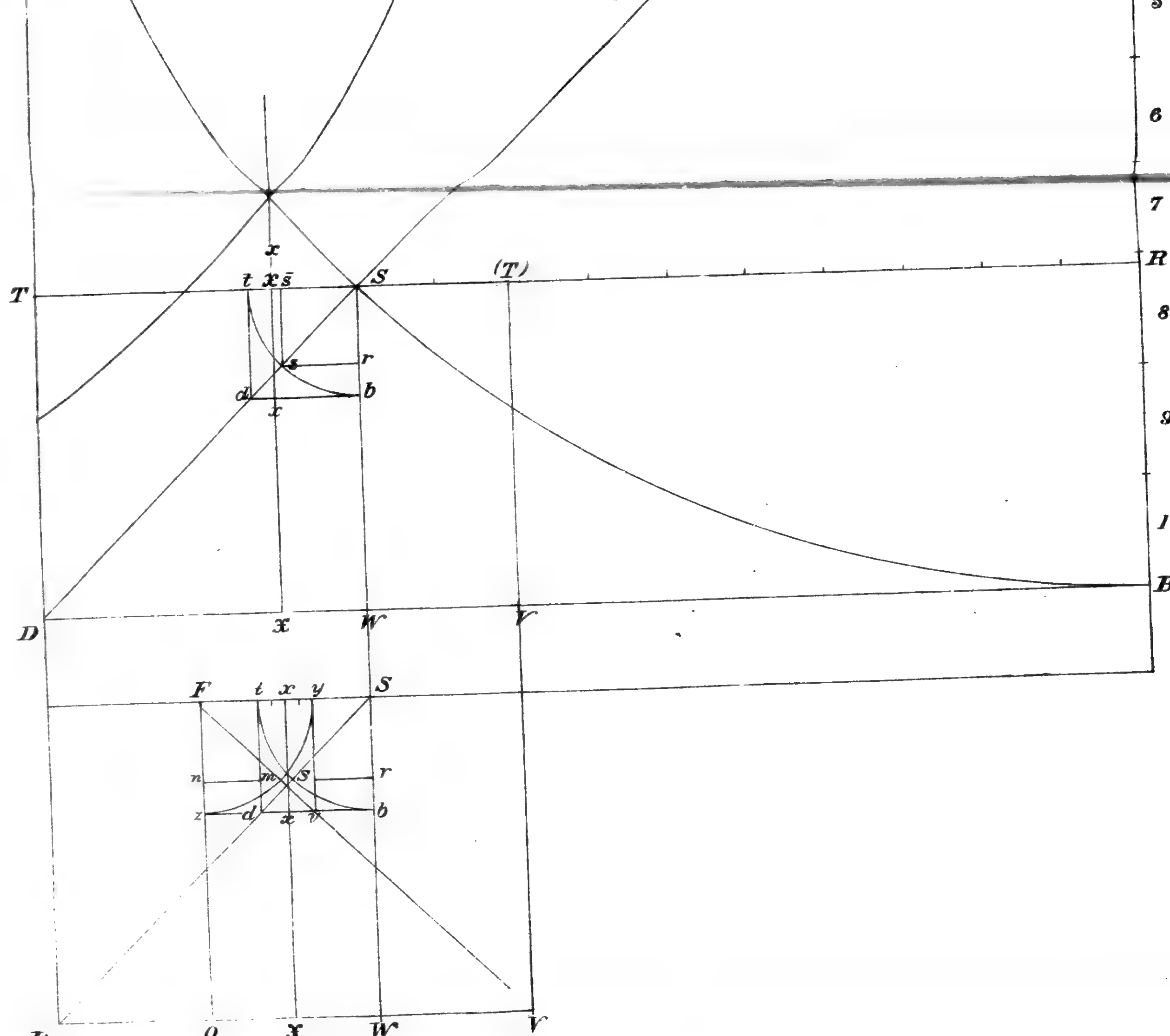


FIG. 12. (b)





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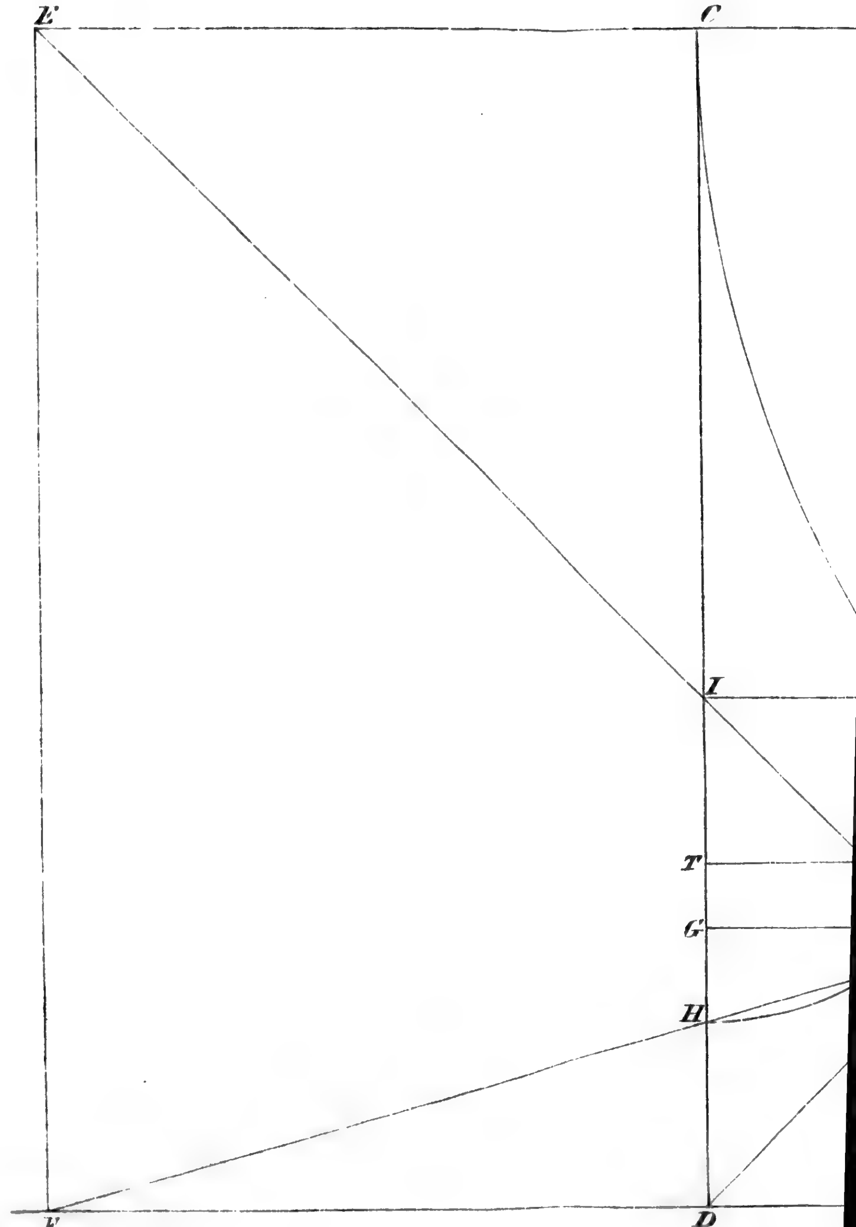
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FIG. 13.

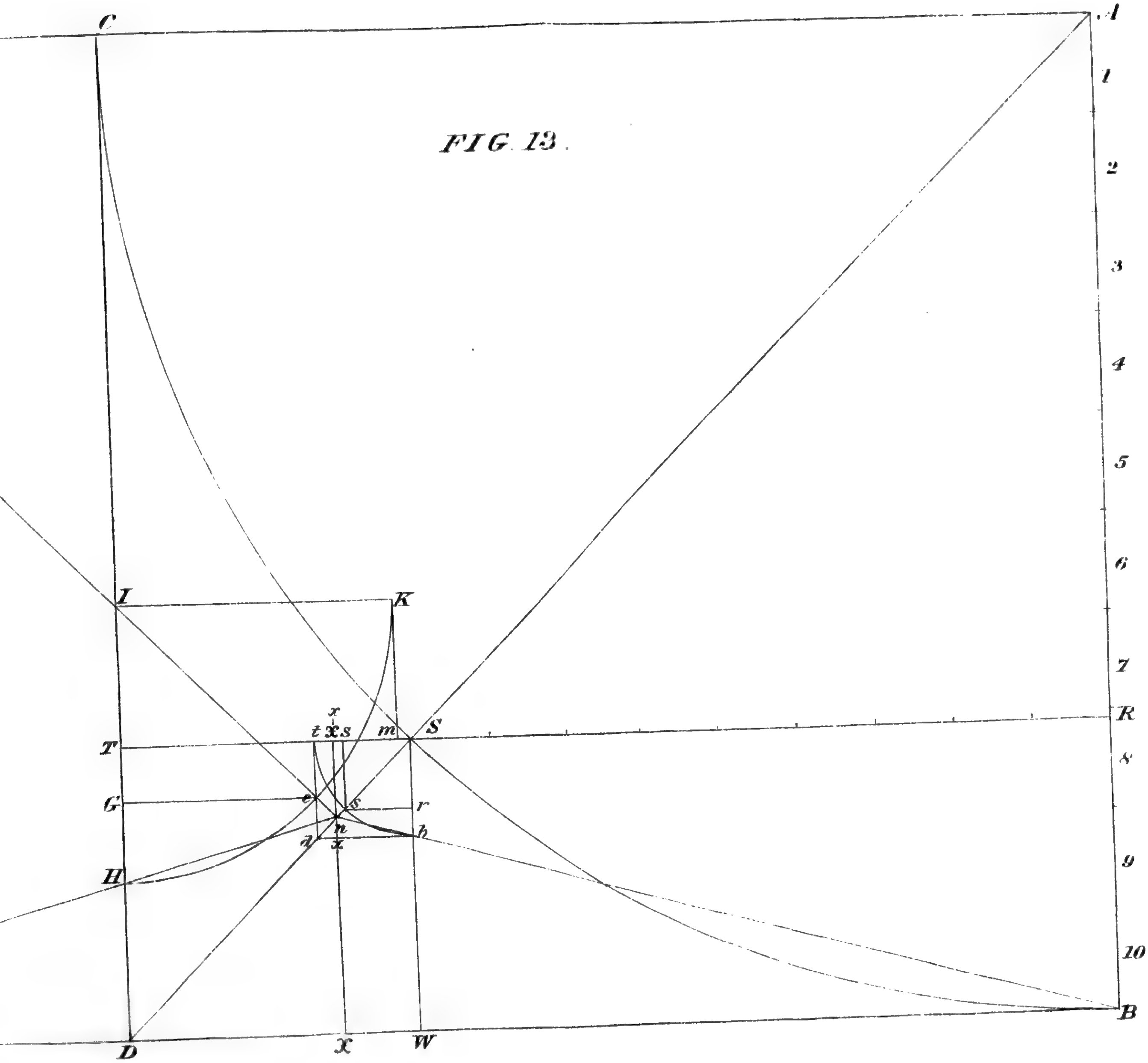
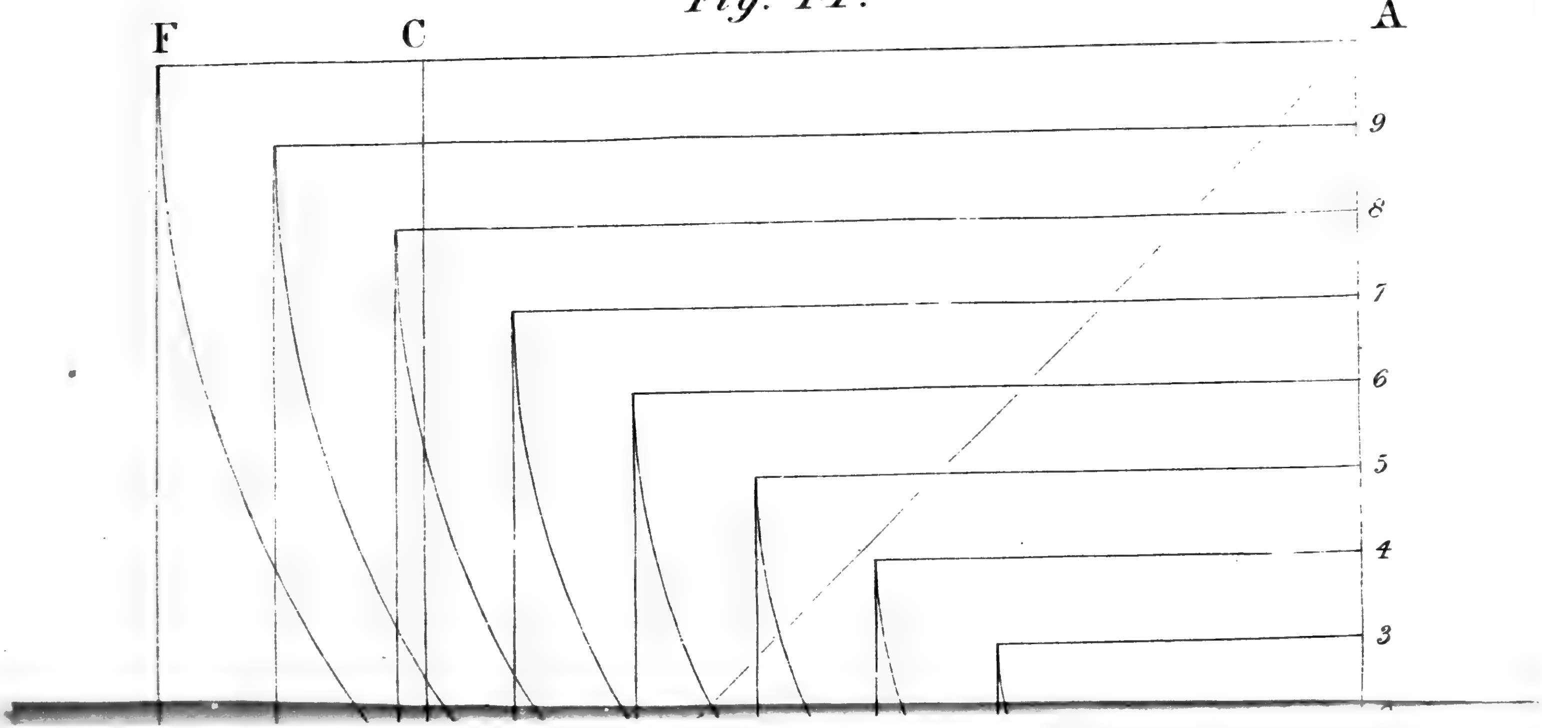


Fig. 14.

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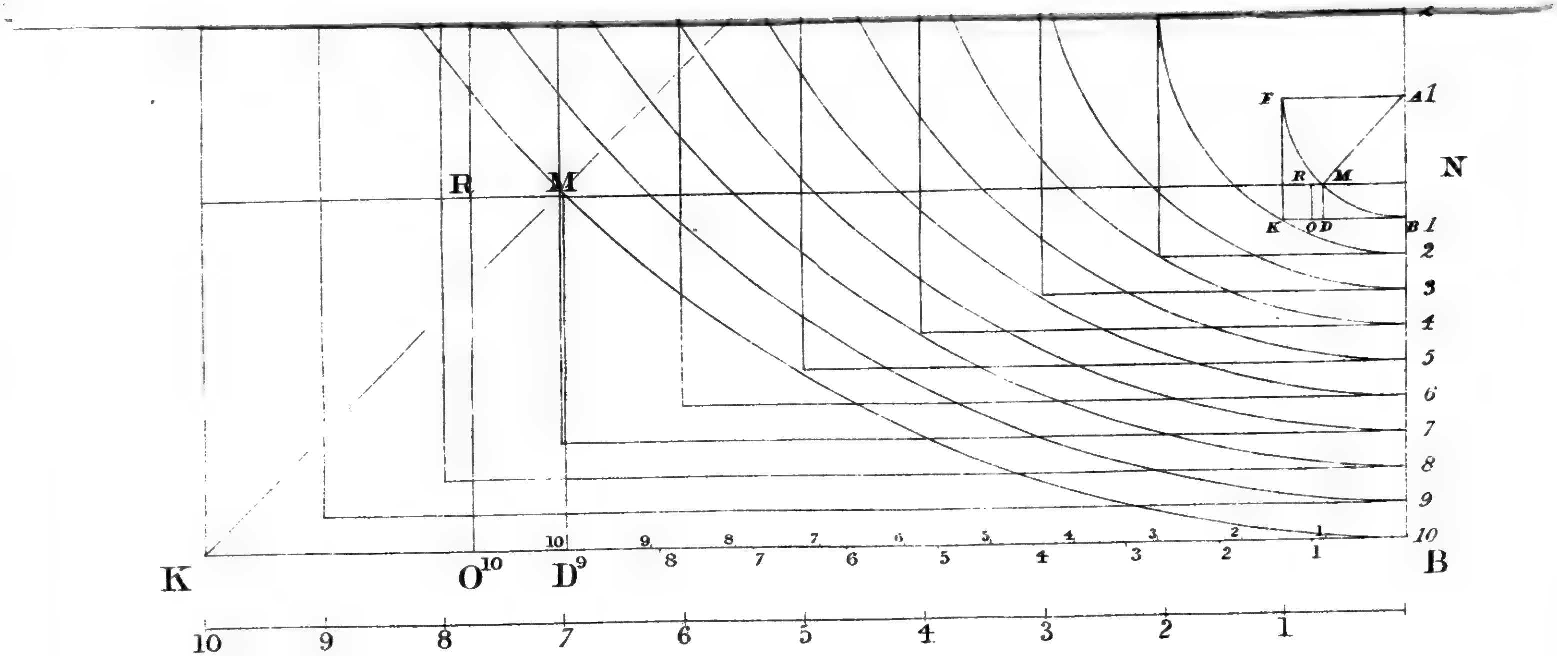
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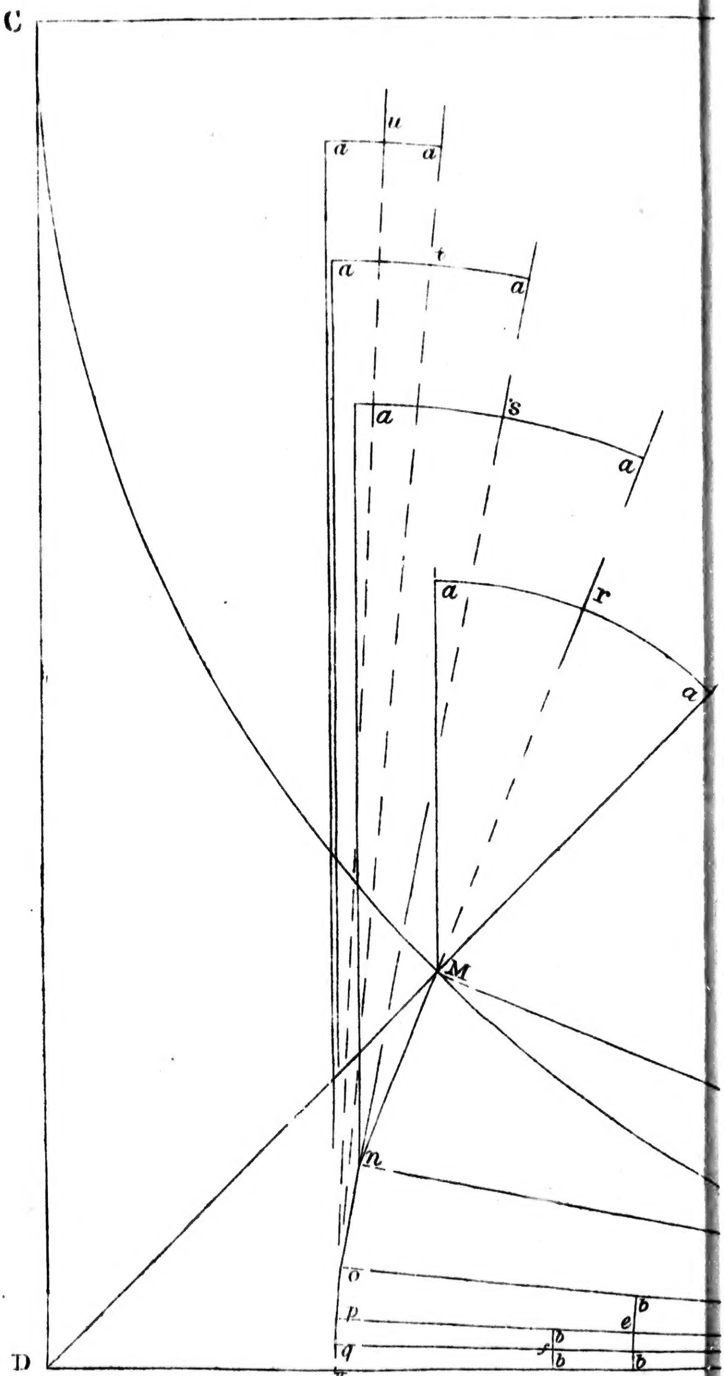

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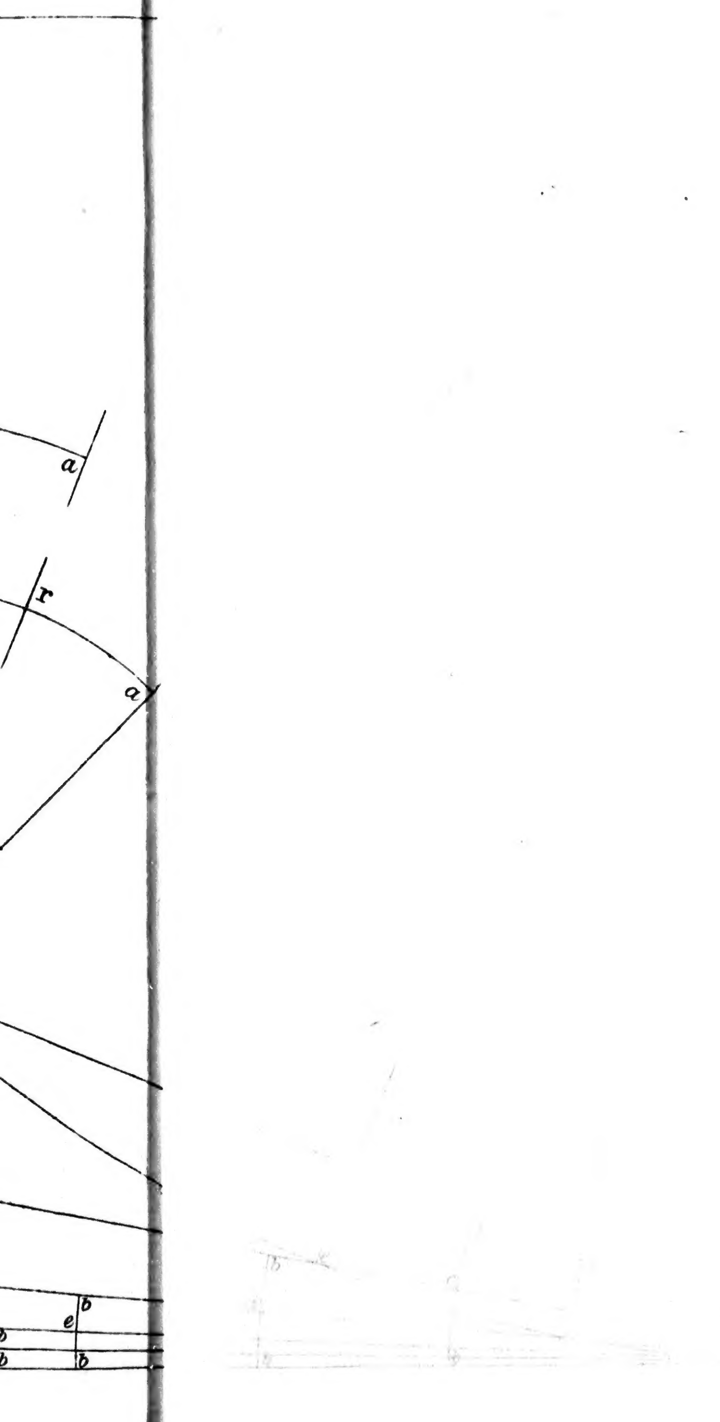

FIG. 20.

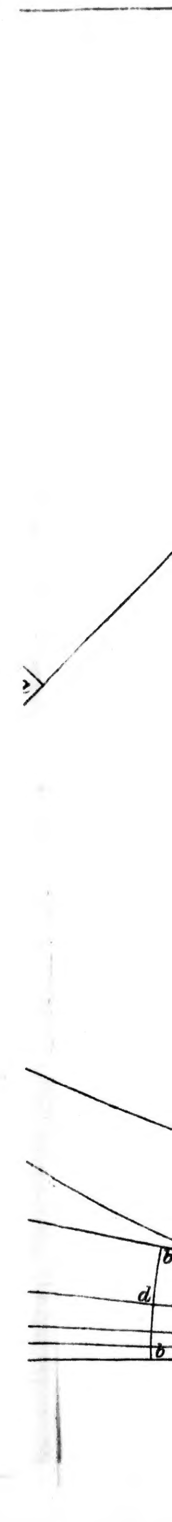

FIG. 20.